10 Derivatives, Part IIb (Leibniz notation)

The notation f' that we've used so far is called the Lagrange notation.[13] However, there is another notation for the derivative in common use. You may have already seen something like $\frac{dy}{dx}$. This is called the Leibniz notation.

The Leibniz notation has many of what Spivak calls "vagaries". It has multiple interpretations– formal and informal. The informal interpretation doesn't map to modern mathematics, but can *sometimes* be useful (while at other times misleading). The full, unambigous Leibniz notation, at least as Spivak defines it, is verbose, so in practice people end up taking liberties with it. As a consequence, its meaning must often be discerned from the context.

This flexibility makes the notation very useful in science and engineering, but also makes it difficult to learn. Spivak chose to standardize on the Lagrange notation to maximize clarity, and banished Leibniz notation to problem sections. But since the Leibniz notation is so common, I take a different approach and explore it here in a dedicated chapter.

10.1 Historical motivation

We start with the historical interpretation, where the notation began. Leibniz didn't know about limits. He thought the derivative is the value of the quotient

$$\frac{f(x+h)-f(x)}{h}$$

when h is "infinitesimally small". He denoted this infinitesimally small quantity of h by dx, and the corresponding difference $f(x+dx)-f(x)$ by $df(x)$. Thus for a given function f the Leibniz notation for its derivative f' is:

$$\frac{df(x)}{dx} = f'$$

Intuitively, we can think of d in a historical context as "delta" or "change". Then we can interpret this notation as Leibniz did– a quotient of a tiny change in $f(x)$ and a tiny change in x. But this explanation comes with two important disclaimers.

First, d is not a value. If it were a value, you could cancel out d's in the numerator and the denomenator. But you can't. Instead think of d as an operator. When applied to $f(x)$ or x, it produces an infinitesimally small quantity. Alternatively you can think of $df(x)$ and dx as one symbol that happens to look like multiplication, but isn't.[14]

[13] Wikipedia claims the notation was invented by Euler and Lagrange only popularized it.

[14] I read somewhere that in his notebooks Leibniz experimented with extending d with a squiggle on top that went over x to indicate that d is not a value, but I haven't been able to verify if that's true.

Second, note that $\frac{df(x)}{dx}$ denotes a function equivalent to f', *not* a value equivalent to $f'(x)$. To denote the image of the derivative function at a we use the following notation:

$$\left.\frac{df(x)}{dx}\right|_{x=a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$$

10.2 Modern interpretation

To summarize, the **full and unambiguous Leibniz notation** in modern interpretation is:

$$\frac{df(x)}{dx} = f' \qquad \text{and} \qquad \left.\frac{df(x)}{dx}\right|_{x=a} = f'(a)$$

Real numbers do not have a notion of infinitesimally small quantities. Thus in a modern interpretation we treat $\frac{df(x)}{dx}$ as a symbol denoting f', *not* as a quotient of numbers. Nothing here is being divided, nothing can be canceled out. In a modern interpretation $\frac{df(x)}{dx}$ is just one thing that *happens to look* like a quotient but isn't, anymore than f' is a quotient.

10.3 Second derivative

A question arises for how to express the second (or nth) derivative in the Leibniz notation. Let $g(x) = \frac{df(x)}{dx}$ (i.e. let g be the first derivative of f). Then it follows that the second derivative in Leibniz notation is $\frac{dg(x)}{dx} = g' = f''$. Substituting the definition of g we get:

$$\frac{d\left(\frac{df(x)}{dx}\right)}{dx} = f''$$

Of course this is too verbose and no one wants to write it this way. This is where the vagaries begin. For convenience people use the usual algebraic rules to get a simpler notation, eventhough formally everything is one symbol and you can't actually do algebra on it:

$$\frac{d\left(\frac{df(x)}{dx}\right)}{dx} = \frac{d^2 f(x)}{dx^2}$$

Two questions arise here.

First, why dx^2? Shouldn't it be $(dx)^2$? One way to answer this question is to remember that dx is one symbol, *not* a multiplication (because d is not a value). And so we're just squaring that one symbol dx, which doesn't require parentheses.

Another probably more honest way to answer this question is to recall that this isn't real algebra– we just use a simularcum of algebra out of convenience. But convenience is a morally flexible thing, and people decided to drop parentheses because they're a pain to write. So $(dx)^2$ became dx^2.

Second, we said before that $df(x)$ can be thought of as one symbol. Then what is this d^2 business? The answer here is the same– we aren't doing real algebra, but a simularcum of algebra out of convenience. We aren't really squaring anything; we're overloading exponentiation to mean "second derivative". The symbol $d^2f(x)$ is again one symbol.

10.4 Liberties and ambiguities

There are a few more liberties people take with the Leibniz notation. Let $f(x) = x^2$. If we want to denote the derivative of f we can do it in two ways:

$$\frac{df(x)}{dx} \qquad \text{or} \qquad \frac{dx^2}{dx}$$

Here $\frac{dx^2}{dx}$ is new, but the meaning should be clear. We're just replacing $f(x)$ in $df(x)$ with the definition of $f(x)$. This is a little confusing because in the particular case of $f(x) = x^2$, it's visually similar to the notation for second derivative. There are no ambiguities here so far– it's just a visual artifact of the notation we have to learn to ignore. But now the liberties come.

Suppose we wanted to state what the derivative of f is. In Lagrange notation we say $f'(a) = 2a$. In Leibniz notation the proper way to say it would be as follows:

$$\left.\frac{df(x)}{dx}\right|_{x=a} = 2a$$

But this is obviously a pain, so people end up taking two liberties. First, everyone drops the vertical line that denotes the application at a. So in practice the form above becomes:

$$\frac{df(x)}{dx} = 2x$$

This shouldn't "compile" because $\frac{df(x)}{dx} = f'$. Thus this statement is equivalent to saying $f' = 2x$, which doesn't make sense. But this is the notation most people use, and you have to get used to it.

Second, people decided that writing $\frac{df(x)}{dx}$ is too painful, and in practice everyone writes $\frac{df}{dx}$. This also shouldn't compile (it would be something like writing $\lim_{x \to a} f$, which also doesn't make sense). But again, it's the notation most people use.

To summarize what we have so far:

$$\left.\frac{df(x)}{dx}\right|_{x=a} = 2a \qquad \text{becomes} \qquad \frac{df}{dx} = 2x$$

10.5 Chain rule

How do we express the chain rule $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$ in Leibniz notation? In the full and unambiguous version the chain rule ought to look like this:

$$\frac{df(g(x))}{dx} = \left.\frac{df(y)}{dy}\right|_{y=g(x)} \cdot \frac{dg(x)}{dx}$$

But, surprise, nobody does it this way. Usually people say that if $y = g(x)$ and $z = f(y)$ then:

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

Let's go through some examples of using this formula, and then see what's going on here. Let $z = \sin y, y = \cos x$. Then

$$\begin{aligned}\frac{dz}{dx} &= \frac{dz}{dy} \cdot \frac{dy}{dx} \\ &= \cos y \cdot (-\sin x) \\ &= -\cos(\cos x) \cdot \sin x\end{aligned}$$

How about $z = \sin u, u = x + x^2$? Well,

$$\begin{aligned}\frac{dz}{dx} &= \frac{dz}{du} \cdot \frac{du}{dx} \\ &= \cos u \cdot (2x + 1) \\ &= \cos(x + x^2) \cdot (2x + 1)\end{aligned}$$

How about a more complicated chain $z = \sin v, v = \cos u, u = \sin x$?

$$\begin{aligned}\frac{dz}{dx} &= \frac{dz}{dv} \cdot \frac{dv}{dx} \\ &= \frac{dz}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx} \\ &= \cos v \cdot (-\sin u) \cdot \cos x \\ &= -\cos(\cos(\sin x) \cdot \sin(\sin x) \cdot \cos x\end{aligned}$$

Now, there are a bunch of notational liberties here:

- $y = \ldots$ implicitly defines a function $y(x)$ which is then used in e.g. $\frac{dy}{dx}$. But y can also be referenced as a value (e.g. "plot y when x is ..."). So the deliniation between functions and the values they take on is blurred.
- dz on the left side of the equations (e.g. in $\frac{dz}{dx}$) denotes $f \circ g$. But dz on the right side of the equations (e.g. in $\frac{dz}{dy}$) denotes f. In other words, the denomenator has a bearing on the meaning of the numerator.

- $\frac{dz}{dy}$ denotes the derivative function, but is also understood to be "an expression involing y" that must be substituted with the value of y in the final answer. E.g. in the first example $\frac{dz}{dy}$ is equal to $\cos y$, and we must then substitute y with $\cos x$.

Despite all these quirks and ambiguities, with some practice we begin to see how easy and useful the Leibniz notation is. In the next section we will refine this understanding as we deal with physical problems involving the derivative.

10.6 Implicit differentiation

Suppose we have an equation for a unit circle $x^2 + y^2 = 1$, and we want to know y changes with changes in x. We will solve this problem in two ways. First, using a "brute force" approach by explicitly solving for y and then differentiating. Second, using a technique called *implicit differentiation* that considerably simplifies the problem.

Brute force approach

With the brute force approach we solve for y and differentiate. Observe that $y^2 = 1 - x^2$, and thus there are two solutions (one for half-circle above the x-axis, and one for half-circle below):

$$y = \sqrt{1 - x^2} \qquad \text{and} \qquad y = -\sqrt{1 - x^2}$$

Differentiating, we get:

$$y' = -\frac{x}{\sqrt{1 - x^2}} = -\frac{x}{y} \qquad \text{and} \qquad y' = -\frac{x}{-\sqrt{1 - x^2}} = -\frac{x}{y}$$

Thus $y' = -\frac{x}{y}$ when $y \neq 0$.

Implicit differentiation approach

We now take a different approach and find a solution without explicitly solving for x. We want to find $\frac{dy}{dx}$. The first thing we'll do is take a derivative of each side of the equations:

$$\begin{aligned} & x^2 + y^2 = 1 \\ & \implies \frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}1 \\ & \implies \frac{d}{dx}x^2 + \frac{d}{dx}y^2 = 0 \end{aligned}$$

Now $\frac{dx^2}{dx} = 2x$ by a straightforward application of differentiation theorem 6. But what about $\frac{dy^2}{dx}$? This would tell us how y^2 changes with changes in x (*not* with changes in y), but how to determine that is not obvious. And so we use the chain rule:[15]

$$
\begin{aligned}
& 2x + \frac{dy^2}{dy} \cdot \frac{dy}{dx} = 0 \\
\implies & 2x + 2y \cdot \frac{dy}{dx} = 0 \\
\implies & \frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}
\end{aligned}
$$

[15]This is very handwavy and I'm running out of steam. Spivak discusses implicit differentiation in his chapter on inverse functions, so I expect to come back to this topic later.