Permutation invariance and the quantum geometry exclusion principle

Andrea Di Biagio

ILQGS 2025-04-29

ÖAW AUSTRIAN ACADEMY OF SCIENCES IQI QISS

Permutation invariance and the quantum geometry exclusion principle

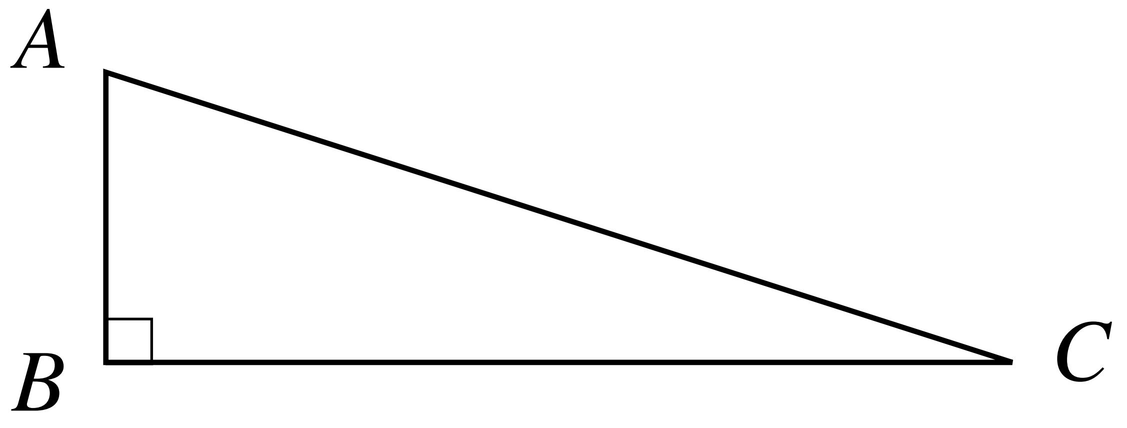
Andrea Di Biagio
ILQGS 2025-04-29

Eugenio Bianchi

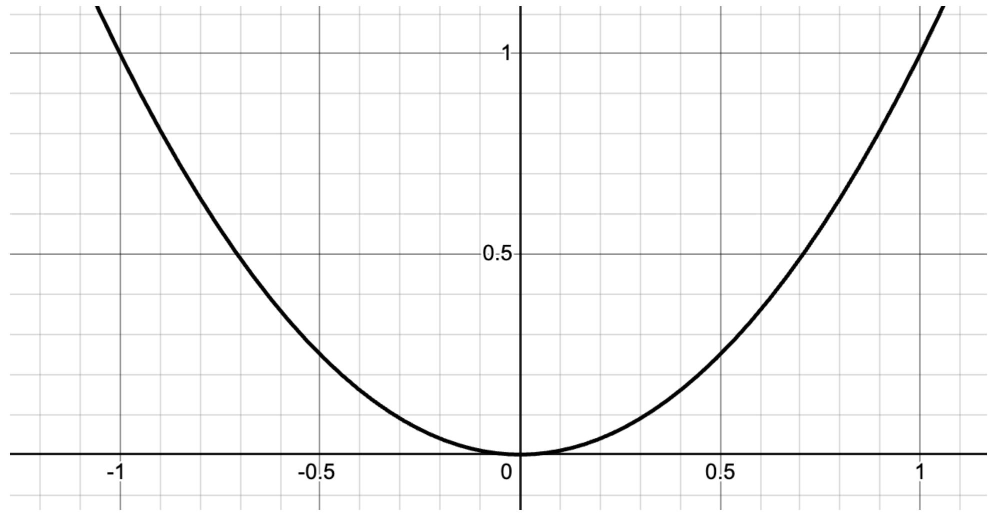
Marios Christodoulou

renaming invariance

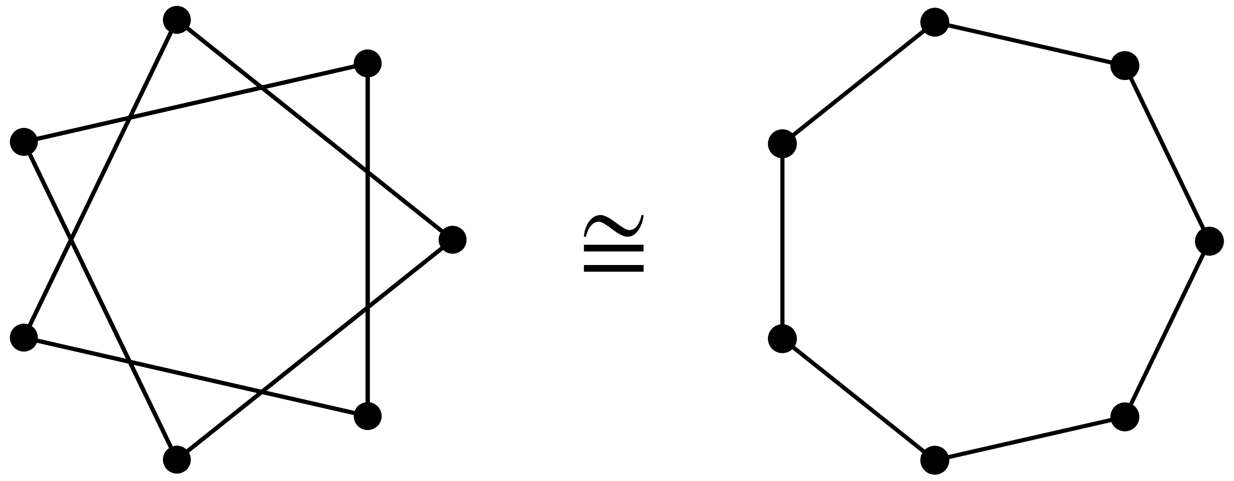
Euclidean geometry



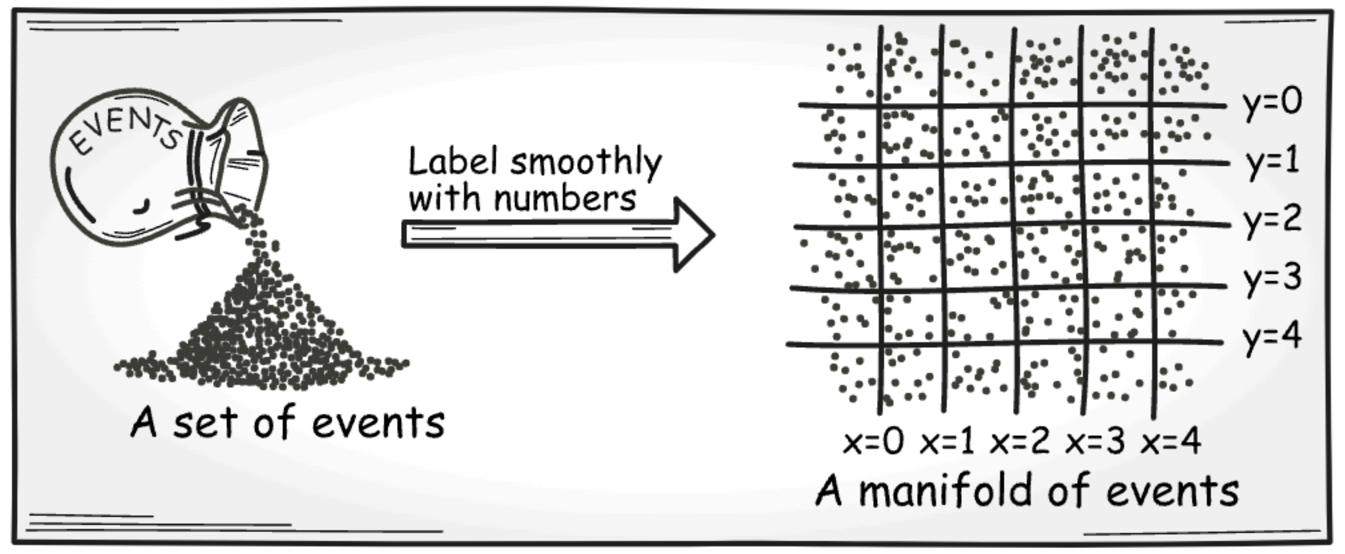
analytic geometry



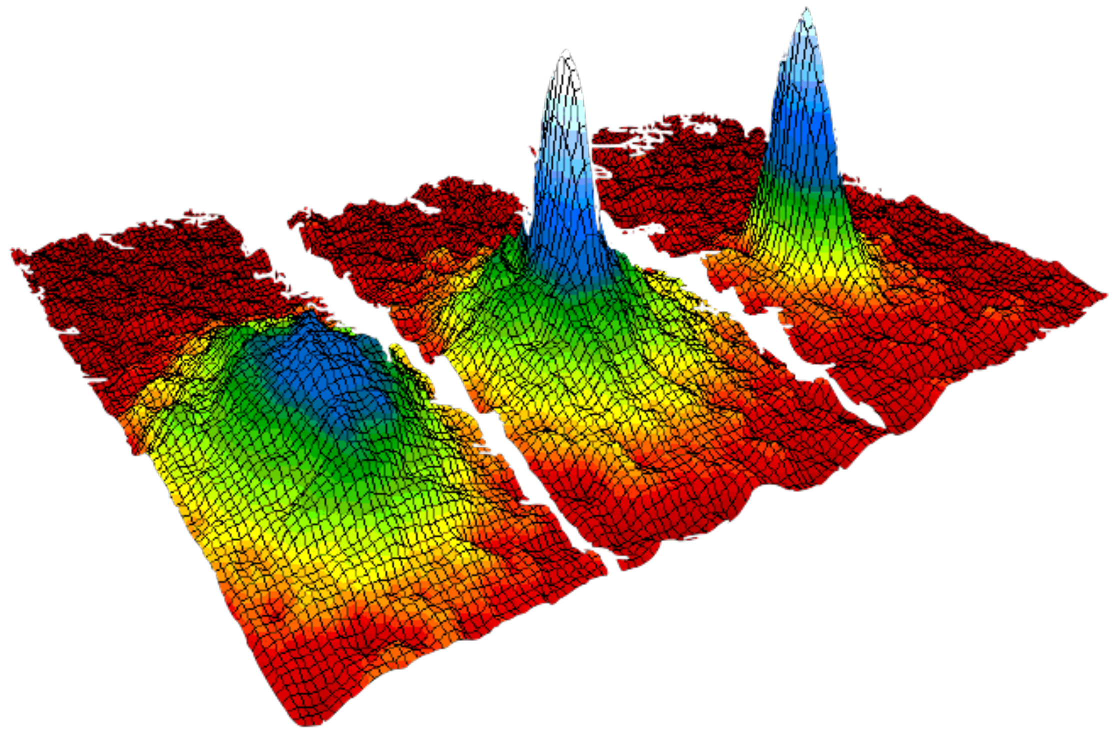
graph theory



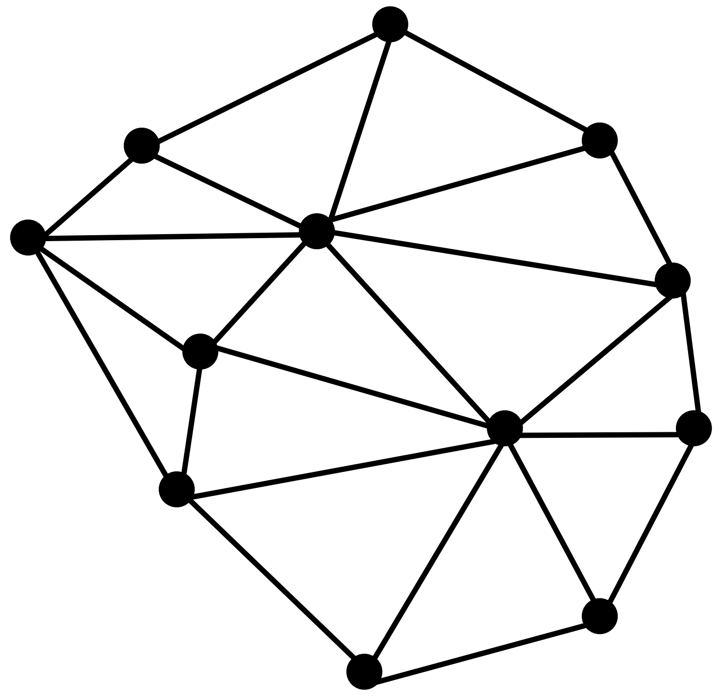
general relativity



condensed matter physics



quantum gravity



[1] Norton, John D., Oliver Pooley, and James Read, "The Hole Argument", *The Stanford Encyclopedia of Philosophy* (Summer 2023 Edition), Edward N. Zalta & Uri Nodelman (eds.), URL = <https://plato.stanford.edu/archives/sum2023/entries/spacetime-holearg/>

[2] NIST/JILA/CU-Boulder

renaming invariance in quantum geometry

the quantum geometry exclusion principle

the geometries of an unlabelled quantum polyhedron different than those of a labelled quantum polyhedron

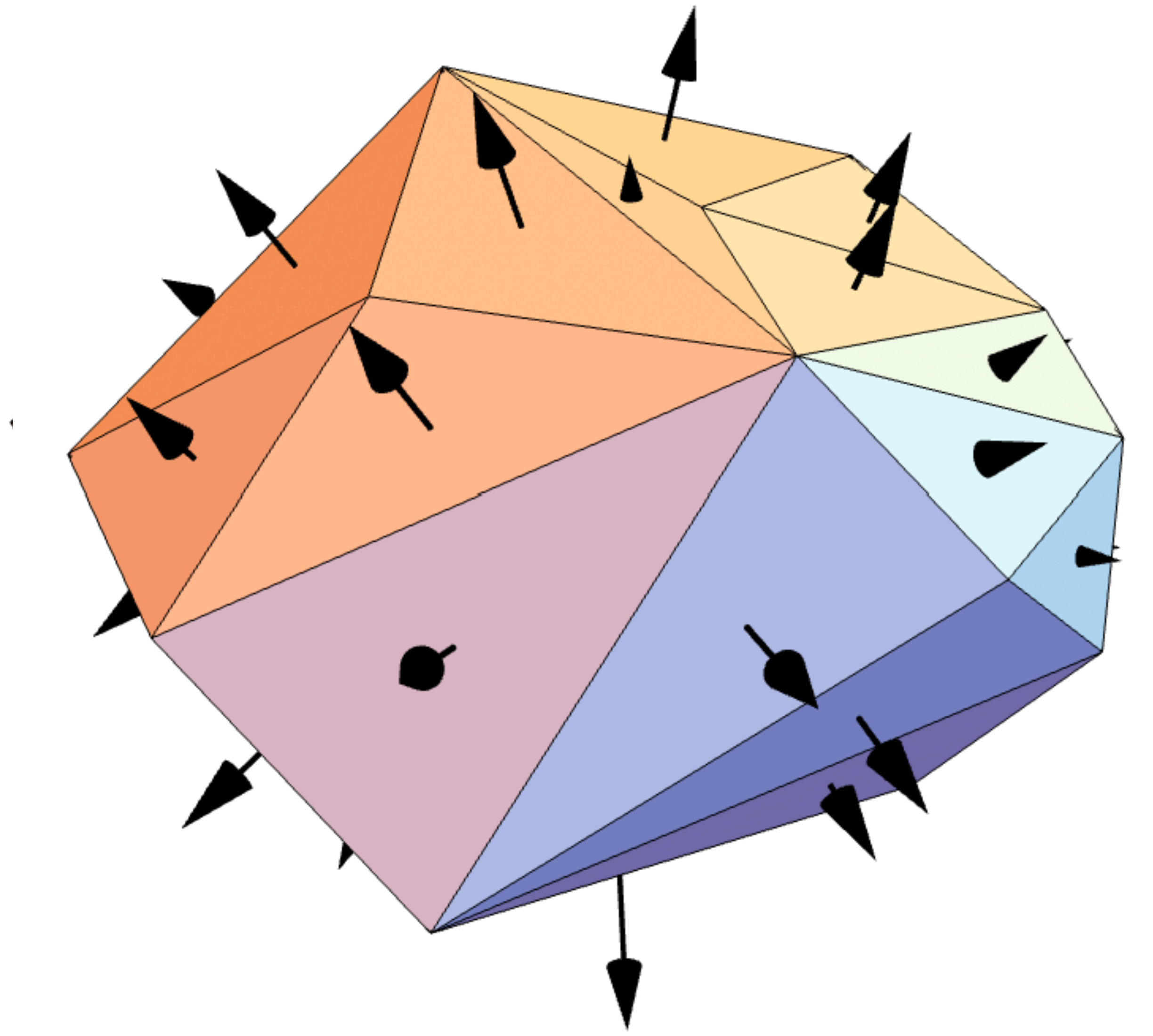
plan

- intro to quantum geometry
- quantum equiarea tetrahedron
  - the QG exclusion principle
  - volume and chirality
  - volume spectrum via Bohr Sommerfeld quantisation
- generalisation to quantum polyhedra

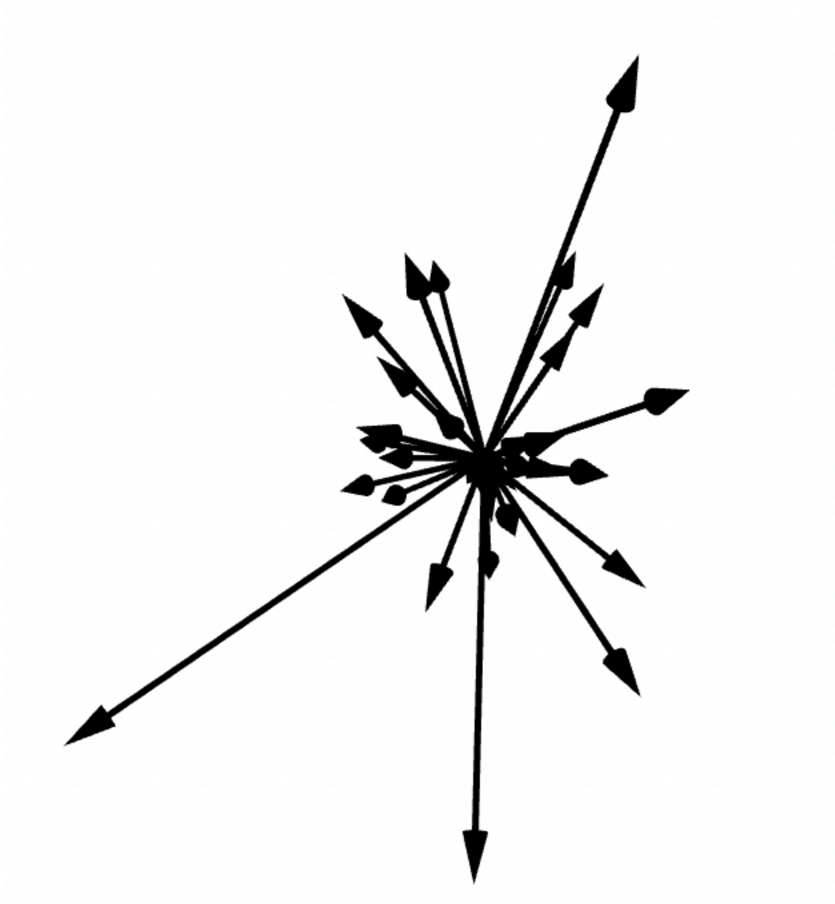
quantum geometry

Minkowski's theorem

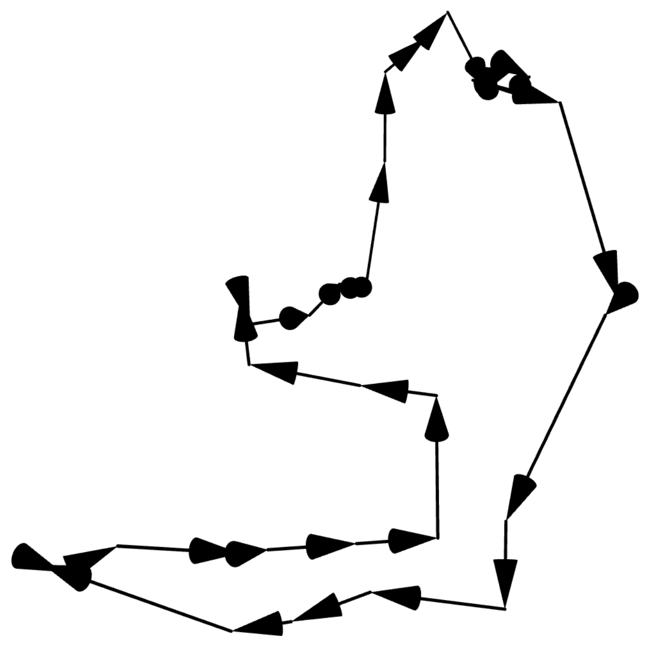
a polyhedron with normals $\vec{A}_a$ $\Longleftrightarrow$ a *set* of vectors $\vec{A}_a$ such that $\sum_a \vec{A}_a = 0$



$\Longleftrightarrow$



:



$$\vec{A}_a = A_a \hat{n}_a$$

$$\sum_a \vec{A}_a = 0$$

[3] Minkowski, H. (1897). Allgemeine Lehrsätze über die konvexen Polyeder. In: Ausgewählte Arbeiten zur Zahlentheorie und zur Geometrie. Teubner-Archiv zur Mathematik, vol 12. Springer, Vienna.
[4] Bianchi, Donà, Speziale (2011). Polyhedra in loop quantum gravity. Phys. Rev. D **83**, 044035. DOI: 10.1103/PhysRevD.83.044035

phase space of polygons

fix a *list* of areas A_a

the space of vectors $\vec{A}_a = A_a \hat{n}_a$ admits a symplectic structure

$$\{A_a^{(i)}, A_b^{(j)}\} = \delta_{ab}\epsilon_{ijk}A_a^{(k)}$$

(N particles on a sphere)

so does the subspace satisfying the constraint $\sum_a \vec{A}_a = 0$

$\Longrightarrow$ Kapovich-Millson phase space of *polygons*

[4] Bianchi, Donà, Speziale (2011). Polyhedra in loop quantum gravity. Phys. Rev. D **83**, 044035. DOI: 10.1103/PhysRevD.83.044035
[5] Kapovich and Millson (1996). The symplectic geometry of polygons in Euclidean space. J. Differential Geom. 44(3): 479-513 DOI: 10.4310/jdg/1214459218

canonical quantisation of the KM phase space

classical phase space:

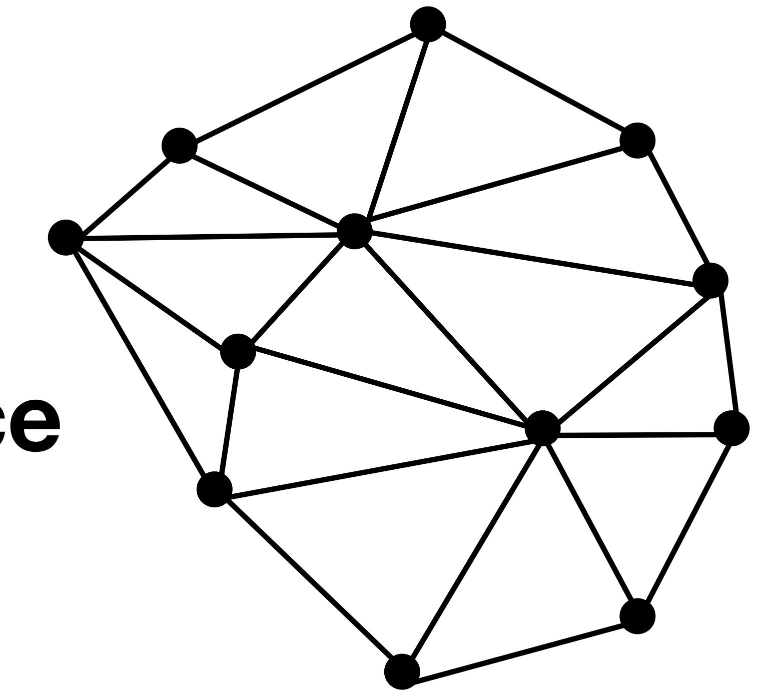
$$\left(\vec{A}_a\right)_{a=1}^{N} \qquad |\vec{A}_a| = A_a \qquad \sum_a \vec{A}_a = 0 \qquad \{A_a^{(i)}, A_b^{(j)}\} = \delta_{ab}\epsilon_{ijk}A_a^{(k)}$$

canonical quantisation:

$$\mathcal{H}^{\text{kin}} = \bigotimes_{a=1}^{N} \mathcal{H}^{(j_a)} \qquad \sqrt{j_a(j_a+1)} = A_a \qquad \sum_a \vec{J}_a = 0 \qquad [J_a^{(i)}, J_b^{(j)}] = i\delta_{ab}\epsilon_{ijk}J_a^{(k)}$$

$$\mathcal{H}^{(0)} = \text{Inv}_{\text{SU}(2)} \bigotimes_{a=1}^{N} \mathcal{H}^{(j_a)}$$

intertwiner space from LQG

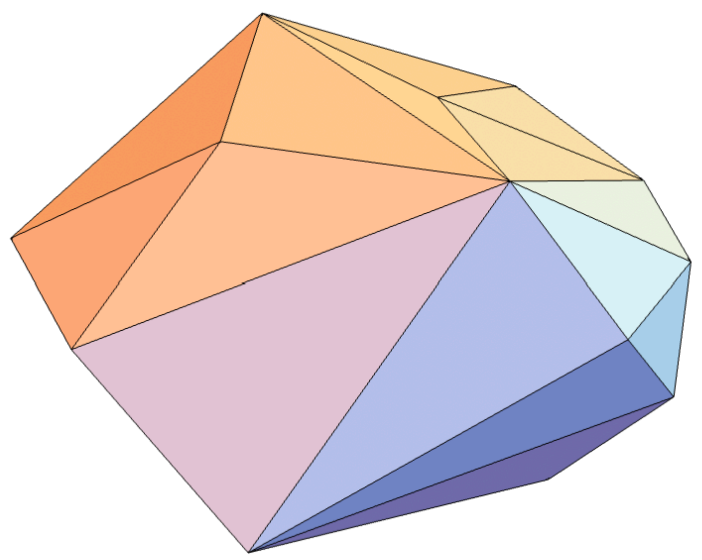


[4] Bianchi, Donà, Speziale (2011). Polyhedra in loop quantum gravity. Phys. Rev. D **83**, 044035. DOI: 10.1103/PhysRevD.83.044035

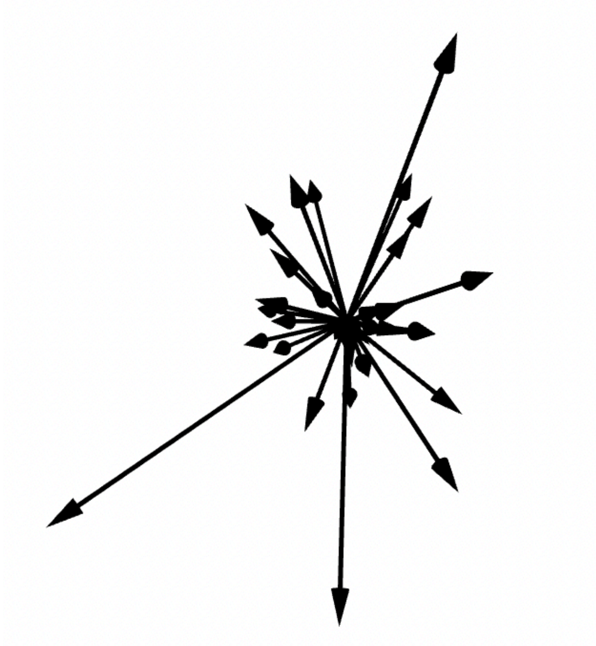
quantum polyhedron?

list of vectors

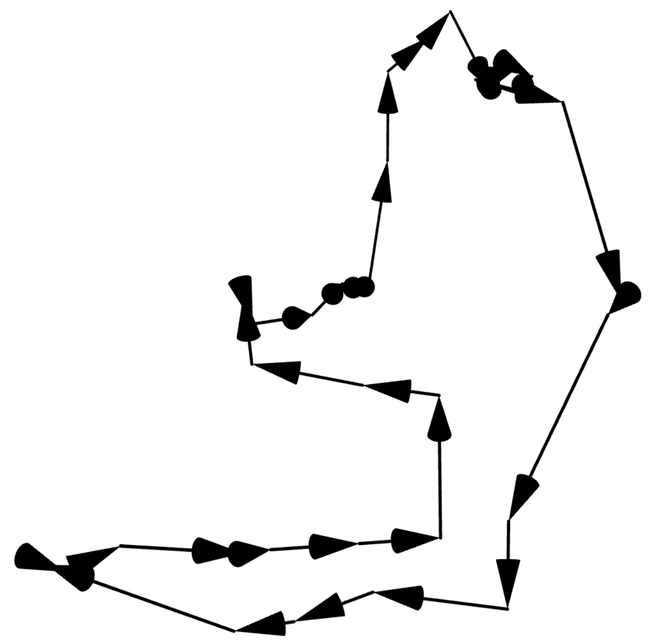
polyhedron



***set* of vectors**



poligon



***labelled* quantum polyhedron**

$$\mathscr{H}^{(0)} = \mathrm{Inv}_{\mathrm{SU}(2)} \bigotimes_{a=1}^{N} \mathscr{H}^{(j_a)}$$

tensor product induces labels

quantum polyhedron

$$\mathscr{H}^{\mathrm{phys}} = \mathrm{Inv}_{\mathrm{SU}(2)\times S_N} \bigotimes_{a=1}^{N} \mathscr{H}^{(j_a)}$$

remove labels by imposing permutation invariance

quantum equiarea tetrahedron

labelled equiarea tetrahedron

kinematical Hilbert space

$$\mathscr{H}^{\text{kin}} = \mathscr{H}^{(j)} \otimes \mathscr{H}^{(j)} \otimes \mathscr{H}^{(j)} \otimes \mathscr{H}^{(j)}$$

$\mathrm{SU}(2)$ **action**

$$U(g) = D^j(g) \otimes D^j(g) \otimes D^j(g) \otimes D^j(g)$$

$\mathrm{SU}(2)$ **invariant subspace**

$$\mathscr{H}^{(0)} = \mathrm{Inv}_{\mathrm{SU}(2)}\, \mathscr{H}^{\text{kin}} = \left\{ |\psi\rangle \in \mathscr{H}^{\text{kin}} \;\middle|\; \vec{J}|\psi\rangle = 0 \right\}$$

constraint

$$\vec{J} = \sum_{a=1}^{4} \vec{J}_a = 0$$

projector

$$P^{(0)} = \int_{\mathrm{SU}(2)} \mathrm{d}g\, U(g)$$

$$\dim \mathscr{H}^{(0)} = \mathrm{tr}\, P^{(0)} = \int_{\mathrm{SU}(2)} \mathrm{d}g\, \mathrm{tr}\, U(g) = \int_{\mathrm{SU}(2)} \mathrm{d}g \prod_{a=1}^{4} \mathrm{tr}\, D^{j_a}(g) = 2j+1$$

permutation group

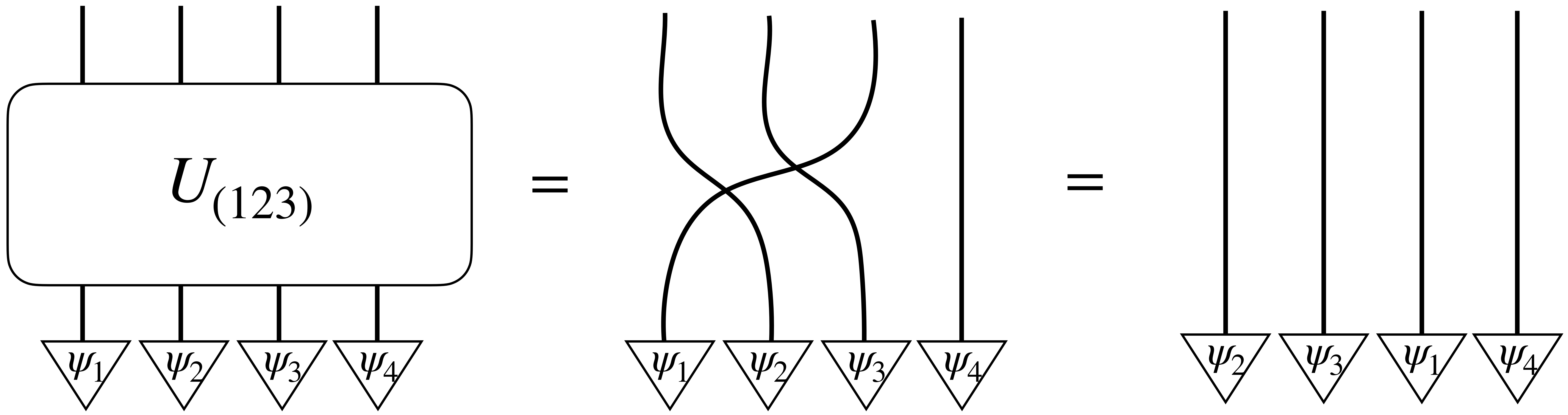
kinematical Hilbert space

$$\mathscr{H}^{\text{kin}} = \mathscr{H}^{(j)} \otimes \mathscr{H}^{(j)} \otimes \mathscr{H}^{(j)} \otimes \mathscr{H}^{(j)}$$

action of the permutation group

$$|\psi\rangle = |\psi_1\rangle|\psi_2\rangle|\psi_3\rangle|\psi_4\rangle$$

$$U_\sigma|\psi\rangle = |\psi_{\sigma(1)}\rangle|\psi_{\sigma(2)}\rangle|\psi_{\sigma(3)}\rangle|\psi_{\sigma(4)}\rangle$$



quantum equiarea tetrahedron

kinematical Hilbert space

$$\mathscr{H}^{\text{kin}} = \mathscr{H}^{(j)} \otimes \mathscr{H}^{(j)} \otimes \mathscr{H}^{(j)} \otimes \mathscr{H}^{(j)}$$

action of $\mathrm{SU}(2)$ and S_4

$$U(g) = D^j(g) \otimes D^j(g) \otimes D^j(g) \otimes D^j(g)$$

$$U_\sigma |\psi\rangle = |\psi_{\sigma(1)}\rangle |\psi_{\sigma(2)}\rangle |\psi_{\sigma(3)}\rangle |\psi_{\sigma(4)}\rangle$$

$$[U(g), U_\sigma] = 0$$

physical Hilbert space

$$\mathscr{H}^{\text{phys}} = \mathrm{Inv}_{\mathrm{SU}(2)\times \mathrm{S}_4} \mathscr{H}^{\text{kin}} = \mathrm{Inv}_{\mathrm{S}_4} \mathscr{H}^{(0)}$$

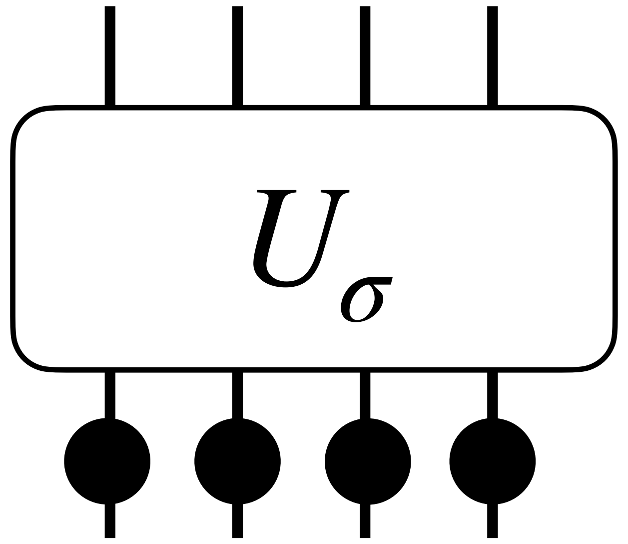
projectors

$$P^{\text{sym}} = \frac{1}{4!} \sum_{\sigma \in \mathrm{S}_4} U_\sigma \qquad P^{(0)} = \int_{\mathrm{SU}(2)} \mathrm{d}g \, U(g)$$

$$\dim \mathscr{H}^{\text{phys}} = \mathrm{tr}\, P^{\text{sym}} P^{(0)} = \frac{1}{4!} \sum_{\sigma \in \mathrm{S}_4} \int_{\mathrm{SU}(2)} \mathrm{d}g \, \mathrm{tr}\, U_\sigma U(g)$$

dimension computation

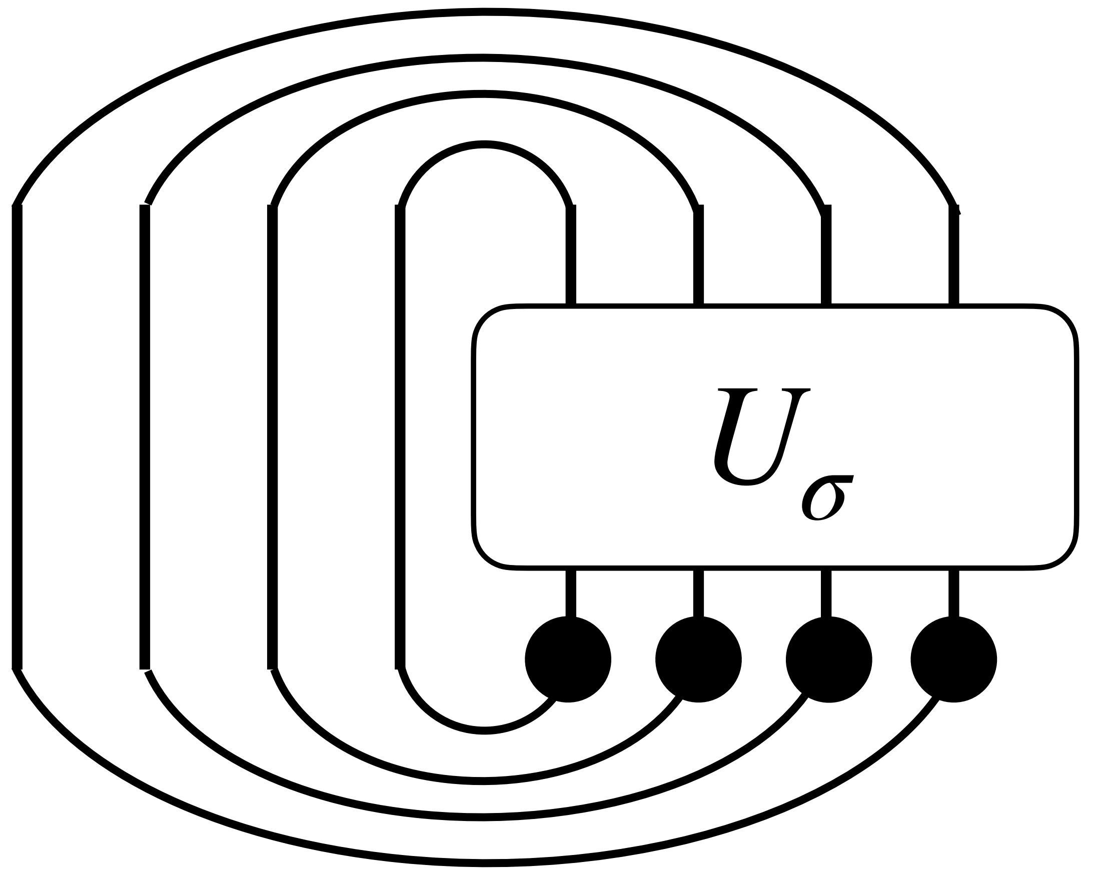
$$\dim \mathscr{H}^{\text{phys}} = \frac{1}{4!} \sum_{\sigma \in S_4} \int_{SU(2)} dg \, \mathrm{tr} \, U_\sigma U(g)$$

$U_\sigma U(g) =$ 

dimension computation

$$\dim \mathscr{H}^{\text{phys}} = \frac{1}{4!} \sum_{\sigma \in S_4} \int_{SU(2)} dg \, \mathrm{tr}\, U_\sigma U(g)$$

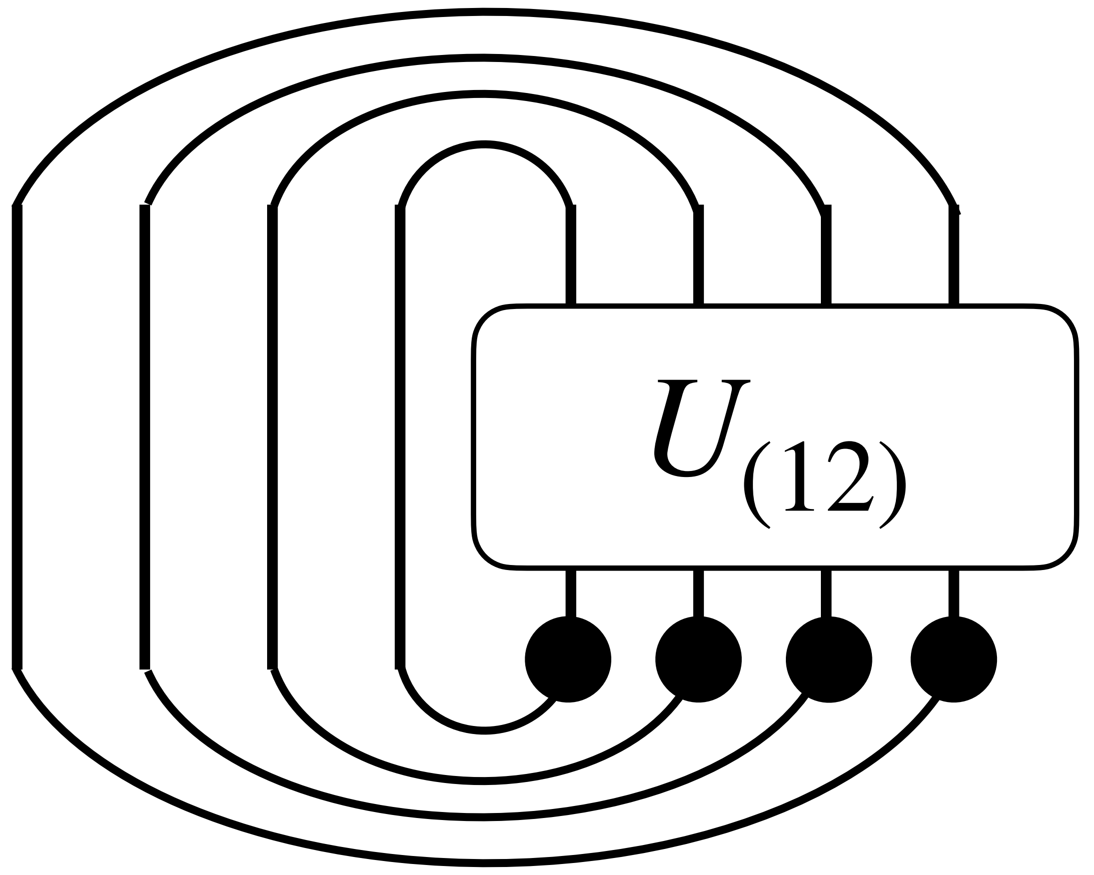
$$\mathrm{tr}\, U_\sigma U(g) =$$



dimension computation

$$\dim \mathscr{H}^{\text{phys}} = \frac{1}{4!} \sum_{\sigma \in \mathrm{S}_4} \int_{\mathrm{SU}(2)} \mathrm{d}g \, \mathrm{tr}\, U_\sigma U(g)$$

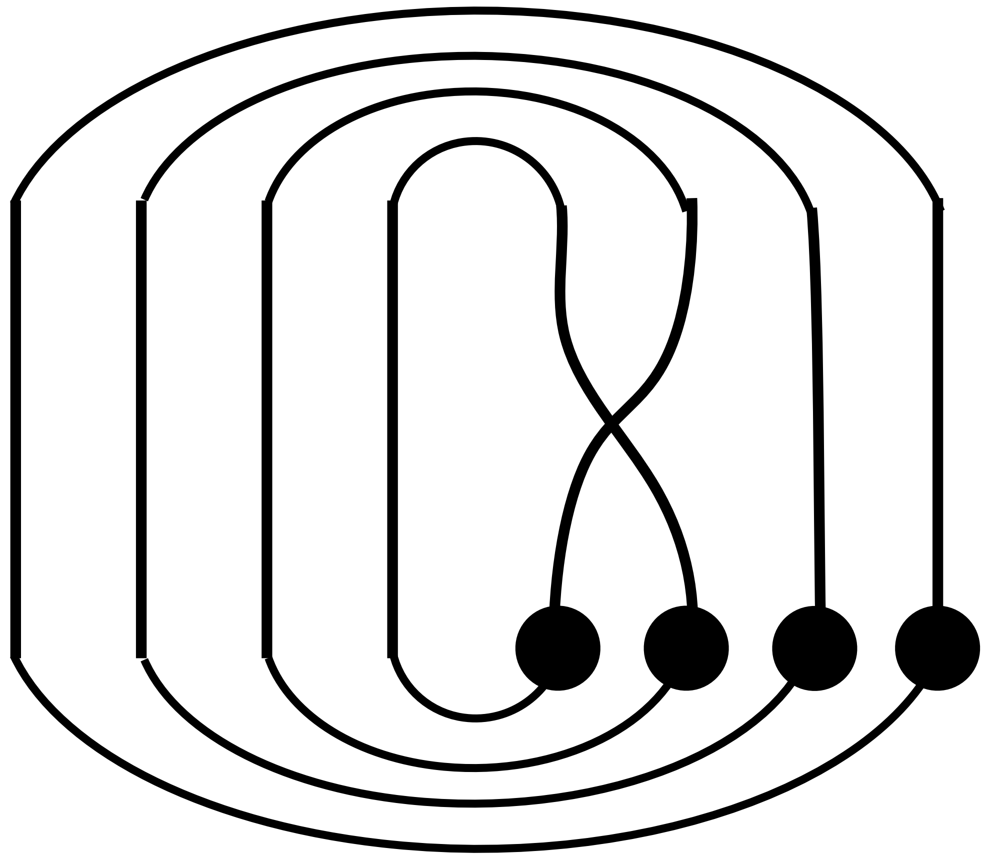
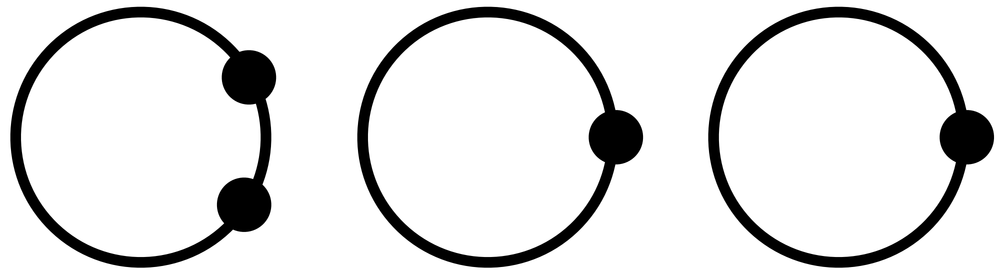
$$\mathrm{tr}\, U_{(12)} U(g) =$$



equiarea tetrahedron

dimension computation

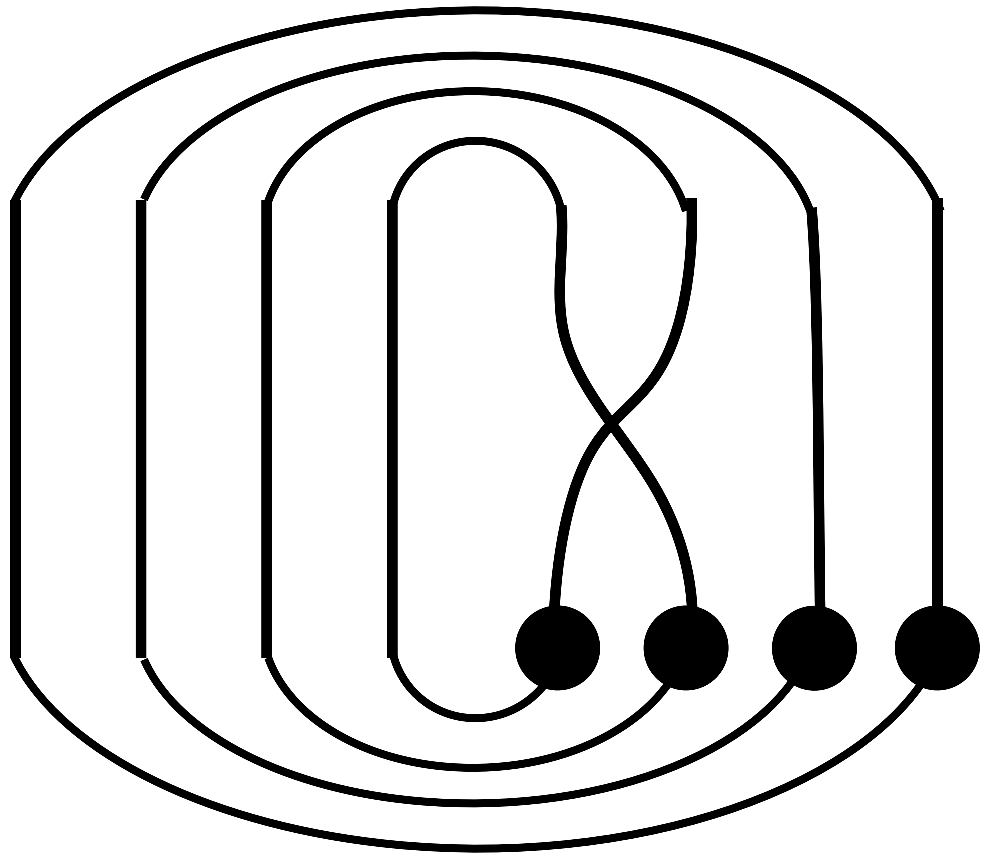
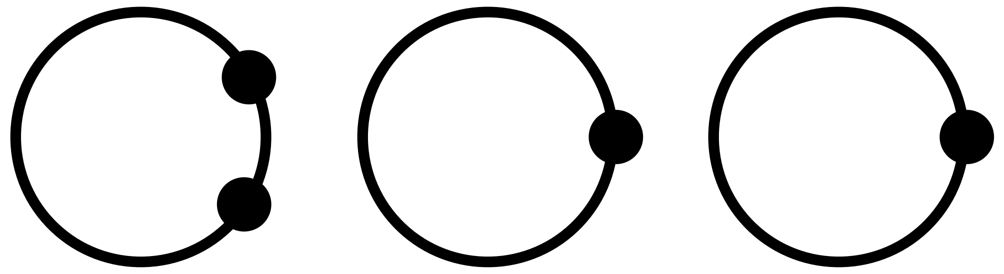
$$\dim \mathscr{H}^{\mathrm{phys}} = \frac{1}{4!} \sum_{\sigma \in \mathrm{S}_4} \int_{\mathrm{SU}(2)} \mathrm{d}g \, \mathrm{tr}\, U_\sigma U(g)$$

$\mathrm{tr}\, U_{(12)} U(g) =$  $=$  $= \mathrm{tr}\, D^j(g^2) \, \left(\mathrm{tr}\, D^j(g)\right)^2$

[6] Coecke, Kissinger (2017) *Picturing Quantum Processes: A First Course in Quantum Theory and Diagrammatic Reasoning*. Cambridge University Press. DOI: 10.1017/9781316219317

dimension computation

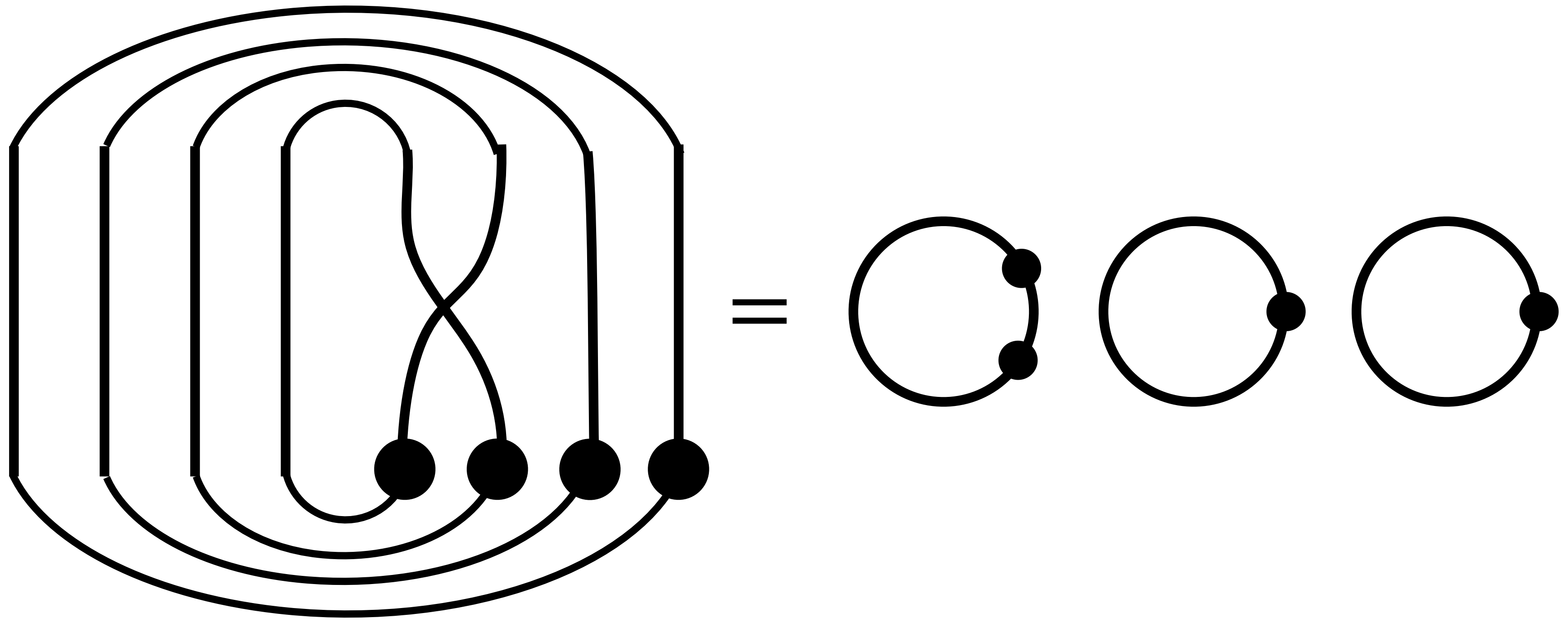
$$\dim \mathscr{H}^{\text{phys}} = \frac{1}{4!} \sum_{\sigma \in \mathrm{S}_4} \int_{\mathrm{SU}(2)} \mathrm{d}g \,\mathrm{tr}\, U_\sigma U(g)$$

$\mathrm{tr}\, U_{(12)} U(g) =$  $=$  $= \mathrm{tr}\, D^j(g^2) \, \left(\mathrm{tr}\, D^j(g)\right)^2$

dimension computation

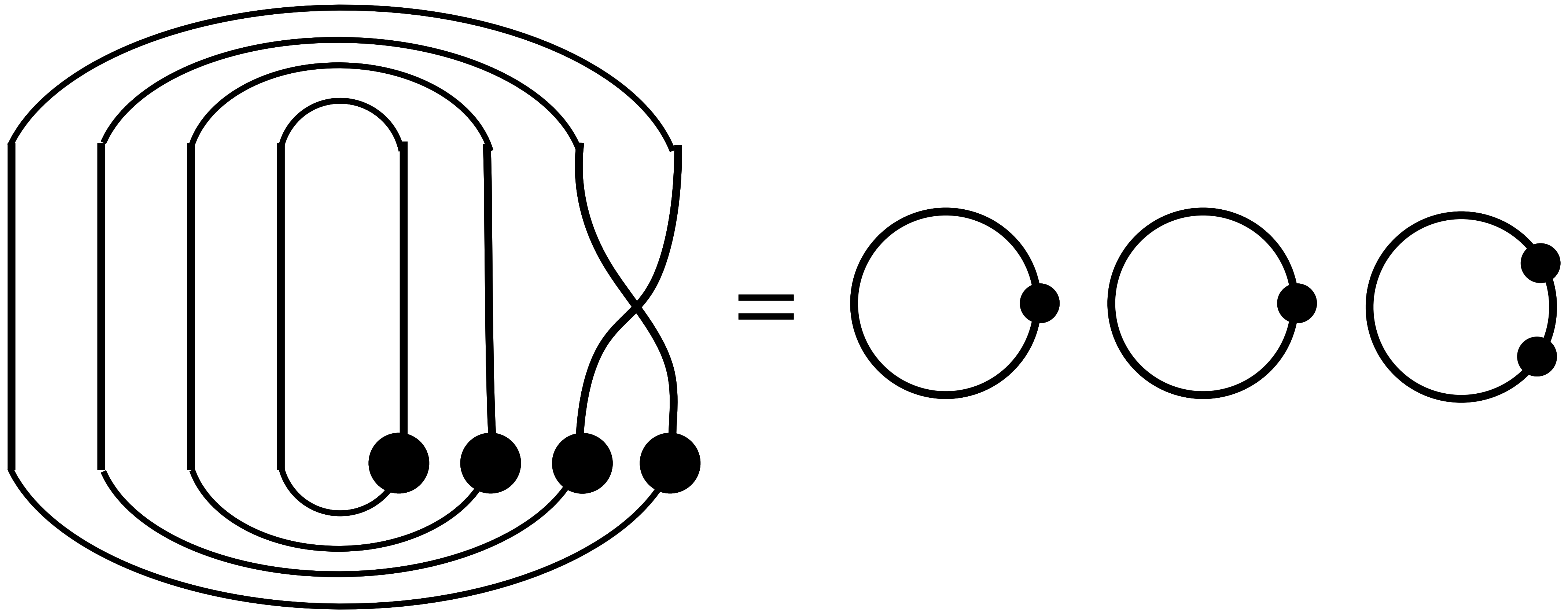
$$\dim \mathscr{H}^{\text{phys}} = \frac{1}{4!} \sum_{\sigma \in S_4} \int_{SU(2)} dg \, \mathrm{tr}\, U_\sigma U(g)$$

$\mathrm{tr}\, U_{(12)} U(g) =$



$= \mathrm{tr}\, D^j(g^2) \, \left(\mathrm{tr}\, D^j(g)\right)^2$

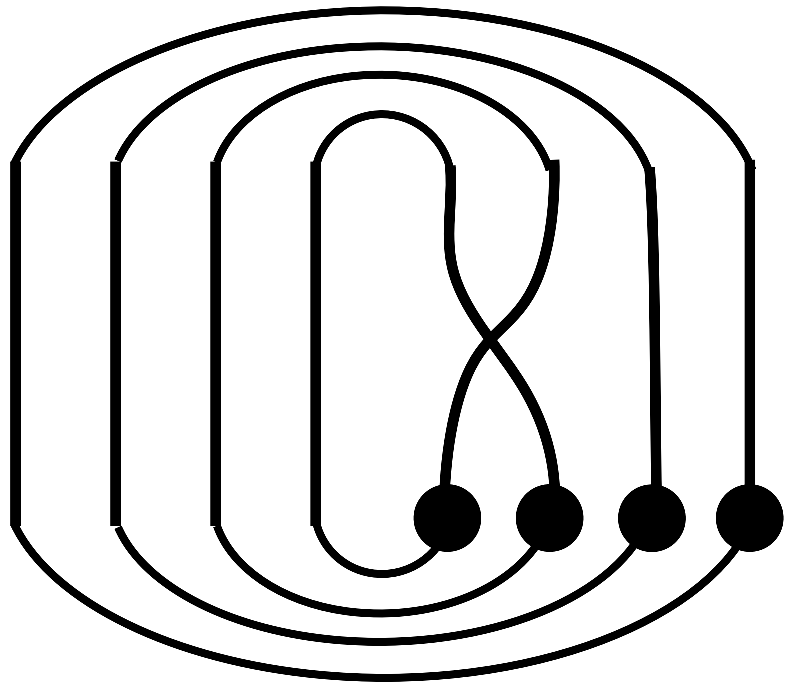
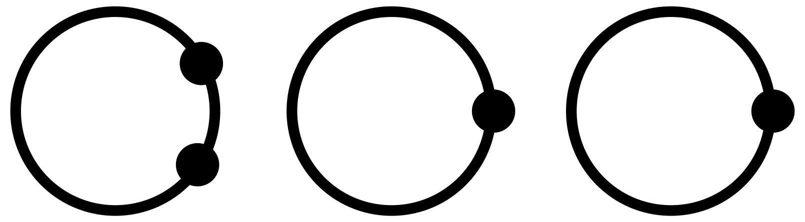
$\mathrm{tr}\, U_{(34)} U(g) =$

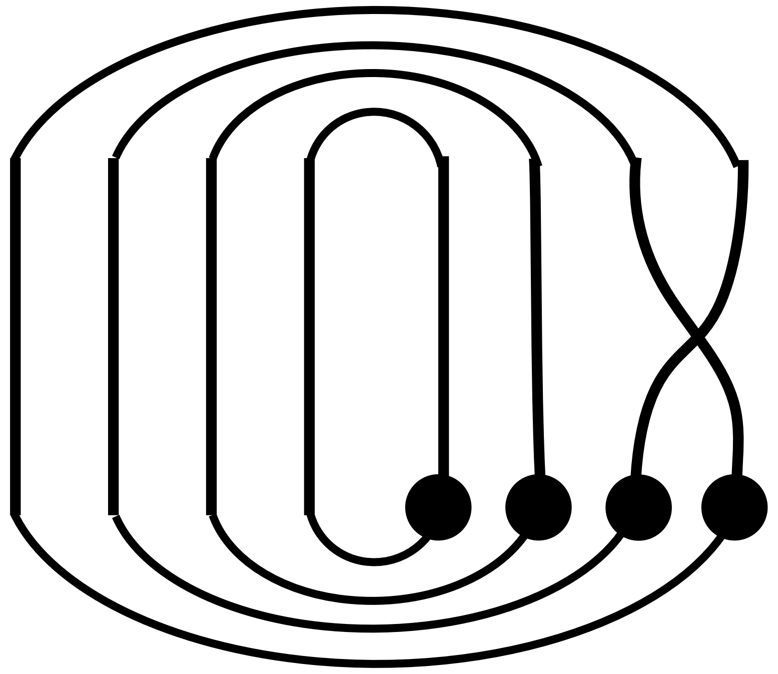
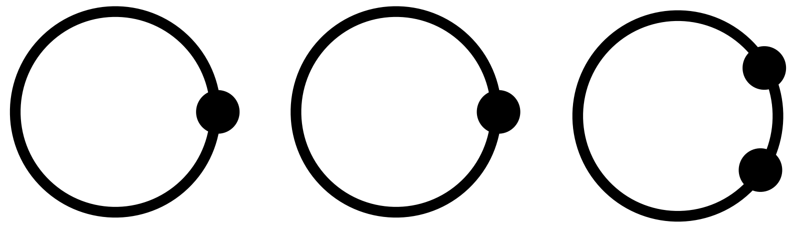


$= \mathrm{tr}\, D^j(g^2) \, \left(\mathrm{tr}\, D^j(g)\right)^2$

dimension computation

$$\dim \mathscr{H}^{\text{phys}} = \frac{1}{4!}\sum_{\sigma\in S_4}\int_{SU(2)} dg\, \mathrm{tr}\, U_\sigma U(g)$$

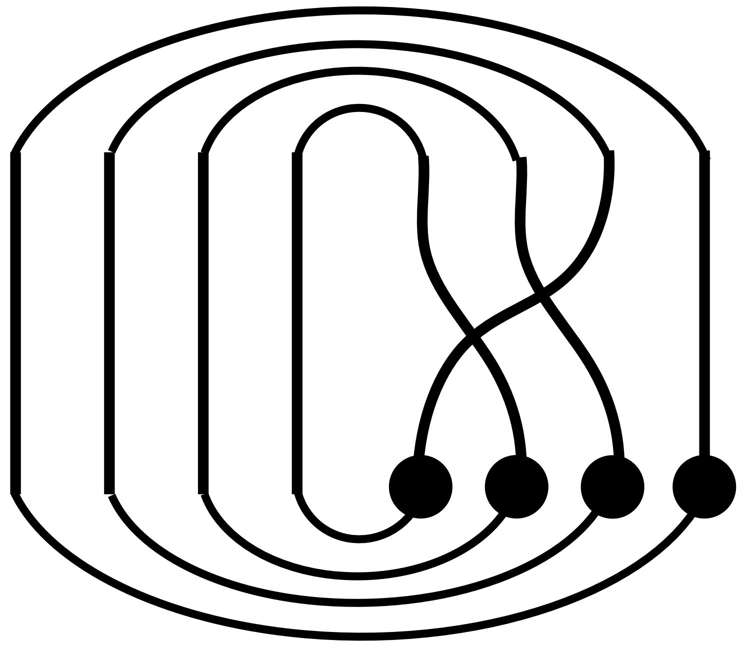
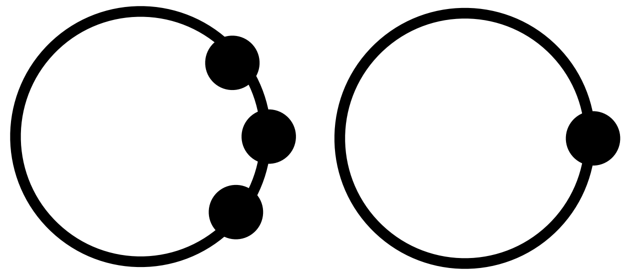
$\mathrm{tr}\, U_{(12)}U(g) =$  $=$  $= \mathrm{tr}\, D^j(g^2)\,\left(\mathrm{tr}\, D^j(g)\right)^2$

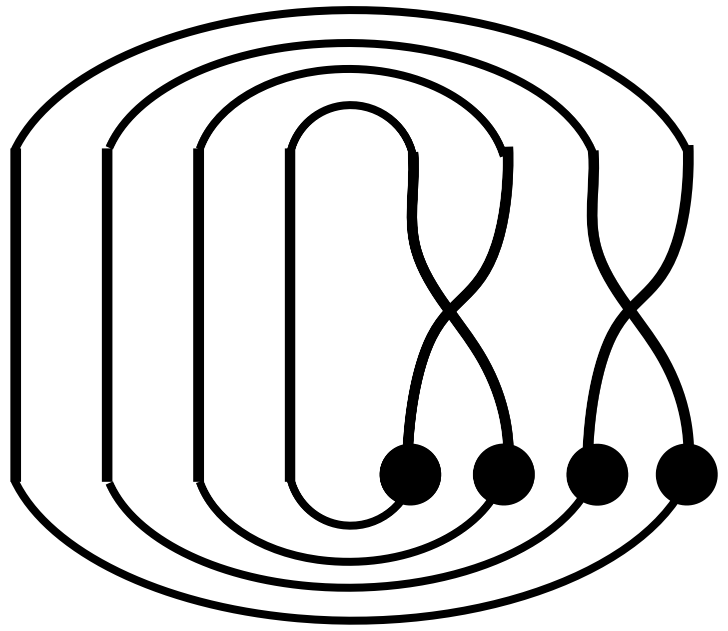
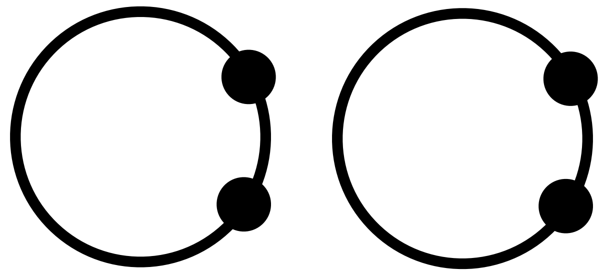
$\mathrm{tr}\, U_{(34)}U(g) =$  $=$  $= \mathrm{tr}\, D^j(g^2)\,\left(\mathrm{tr}\, D^j(g)\right)^2$

depends only on *cycle structure* of σ : **one 2-cycle** , **two 1-cycles**

dimension computation

$$\dim \mathscr{H}^{\text{phys}} = \frac{1}{4!} \sum_{\sigma \in S_4} \int_{SU(2)} dg \, \mathrm{tr}\, U_\sigma U(g)$$

$\mathrm{tr}\, U_{(123)} U(g) =$  $=$  $= \mathrm{tr}\, D^j(g^3)\, \mathrm{tr}\, D^j(g)$

$\mathrm{tr}\, U_{(12)(34)} U(g) =$  $=$  $= \left[\mathrm{tr}\, D^j(g^2)\right]^2$

dimension of the physical Hilbert space

can be computed explicitly!

$$\dim \mathscr{H}^{\text{phys}} = \frac{1}{4!} \sum_{\sigma \in S_4} \int_{SU(2)} dg \, \text{tr}\, U_\sigma U(g) = \frac{1}{4!} \sum_{\lambda \vdash 4} C_\lambda \int_{SU(2)} dg \prod_{k=1}^{4} \left[\text{tr}\, D^j(g^k) \right]^{\mu_\lambda(k)}$$

trace depends only on the cycle structure of σ

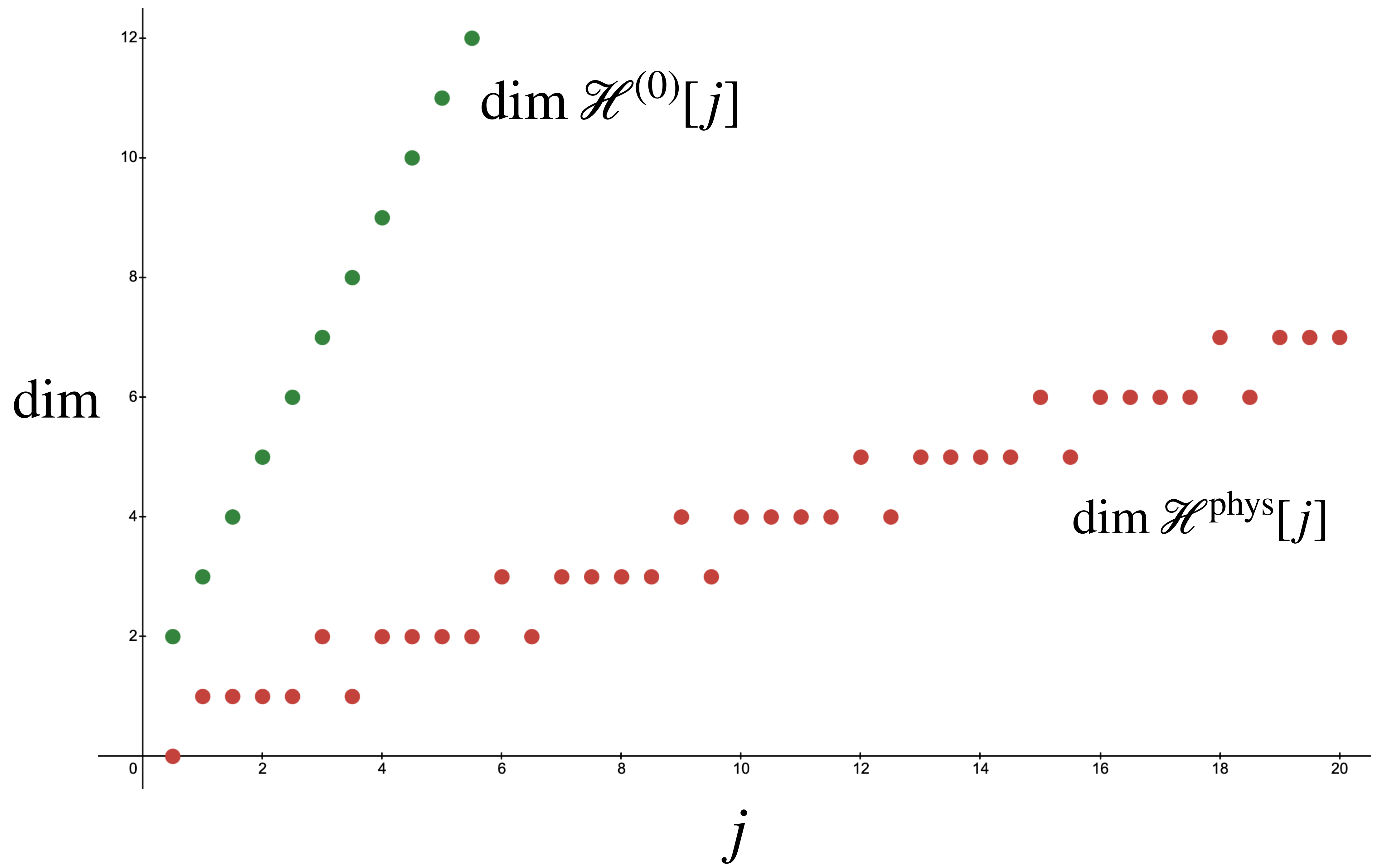
sum over elements becomes sum over equivalence classes (ie cycle structures)

in each class λ, there are $\mu_\lambda(k)$ cycles of length k, each contributing a factor of $\text{tr}\, D^j(g^k)$

dimension of the physical Hilbert space

$$\dim \mathscr{H}^{\text{phys}}[j] = \dim \text{Inv}_{SU(2)\times S_4} \mathscr{H}^{\text{kin}} = \frac{2j - 1 + 3\text{Mod}_2(2j-1) + 2\text{Mod}_3(2j-1)}{6} \sim \frac{1}{3}j$$

$$\dim \mathscr{H}^{(0)}[j] = \dim \text{Inv}_{SU(2)} \mathscr{H}^{\text{kin}} = 2j + 1$$



83% of the states of the quantum tetrahedron excluded by permutation invariance

the quantum geometry exclusion principle

the quantum geometry exclusion principle

imposing renaming invariance reduces the number of states of the quantum polyhedron

a quantum polyhedron with unlabelled faces has fewer configurations than a polyhedron with labelled faces

quantum equiarea tetrahedron

volume and chirality

volume operator

volume operator
for equiarea tetrahedron

$$V = \frac{\sqrt{2}}{3}\sqrt{|\vec{J}_1 \cdot (\vec{J}_2 \times \vec{J}_3)|}$$

since 5/6 of states in $\mathscr{H}^{(0)}$ are lost when imposing permutation invariance,
the spectrum has to be modified

permutation invariant on $\mathscr{H}^{(0)}$ $\quad U_\sigma^\dagger V U_\sigma = V \quad \Longrightarrow$ eigenspaces carry a representation of S_4

$$\mathscr{H}^{(0)} = \bigoplus_v \mathscr{H}_v^{(0)}$$

[8] Bianchi and Haggard (2012) Bohr-Sommerfeld quantization of space Phys. Rev. D **86**, 124010 DOI: 10.1103/PhysRevD.86.124010

signed volume operator

$$\mathscr{H}^{(0)} = \bigoplus_v \mathscr{H}_v^{(0)}$$

signed **volume operator**

$$Q = \frac{2}{9} \vec{J}_1 \cdot (\vec{J}_2 \times \vec{J}_3)$$

$$V = \sqrt{|Q|}$$

"volume + chirality"

eigenstates

$$Q | \pm v\rangle = \pm v^2 | \pm v\rangle$$

$$V | \pm v\rangle = v | \pm v\rangle$$

alternating representation

$$U_\sigma^\dagger Q U_\sigma = \mathrm{sign}(\sigma) Q \quad \Longrightarrow \quad U_\sigma | \pm v\rangle \propto | \pm \mathrm{sign}(\sigma) v\rangle$$

volume eigenspaces carry nontrivial 2d reps of S_4

[8] Bianchi and Haggard (2012) Bohr-Sommerfeld quantization of space Phys. Rev. D **86**, 124010 DOI: 10.1103/PhysRevD.86.124010

signed volume operator

$$\mathcal{H}^{(0)} = \bigoplus_{v} \mathcal{H}_v^{(0)}$$

$v = 0 \implies \dim \mathcal{H}_v^{(0)} = 1$ **invariant subspace**

$v > 0 \implies \dim \mathcal{H}_v^{(0)} = 2$ **2d representation**

when $v > 0$ two options:

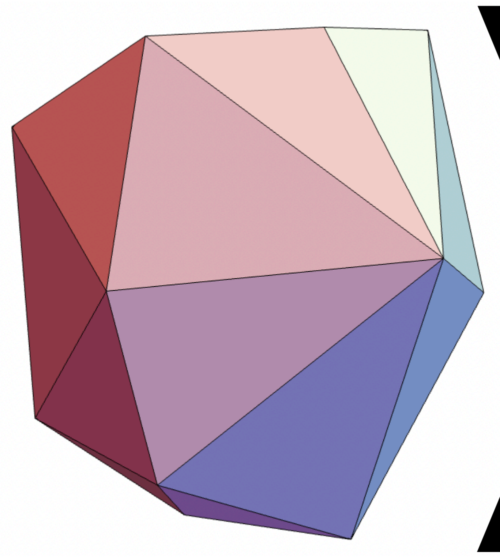
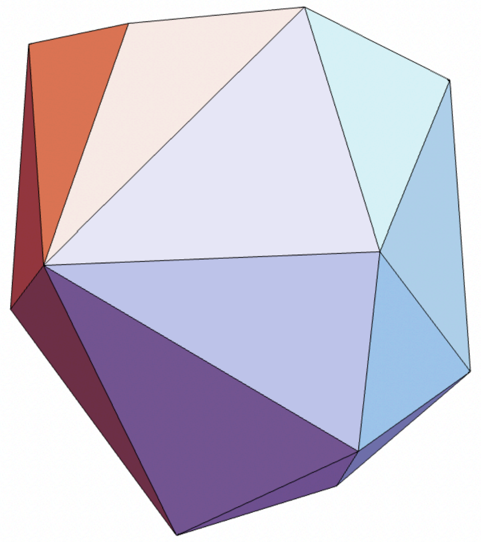
$\phi_\pm(\sigma) \in \left\{0, \frac{2\pi}{3}, -\frac{2\pi}{3}\right\}$

$\mathcal{H}_v^{(0)} \cong \mathcal{H}^{[2,2]}$ **2d irrep of S_4** $\implies$ **v excluded** $U_\sigma|\pm v\rangle = e^{i\phi_\pm(\sigma)}|\pm \mathrm{sign}(\sigma) v\rangle$

$\mathcal{H}_v^{(0)} \cong \mathcal{H}^{[1^4]} \oplus \mathcal{H}^{[4]}$ **reducible** $\implies$ **v allowed** $U_\sigma|\pm v\rangle = |\pm \mathrm{sign}(\sigma) v\rangle$

$$|v^{\mathrm{phys}}\rangle = \frac{1}{\sqrt{2}}|+v\rangle + \frac{1}{\sqrt{2}}|-v\rangle$$

volume and chirality

$|\nu^{\mathrm{phys}}\rangle = |$  $\rangle + |$  $\rangle$

permutation invariant states have indefinite chirality

chirality relies on an ordering of the faces

or **an orientation of the ambient space (but no ambient space in quantum geometry)**

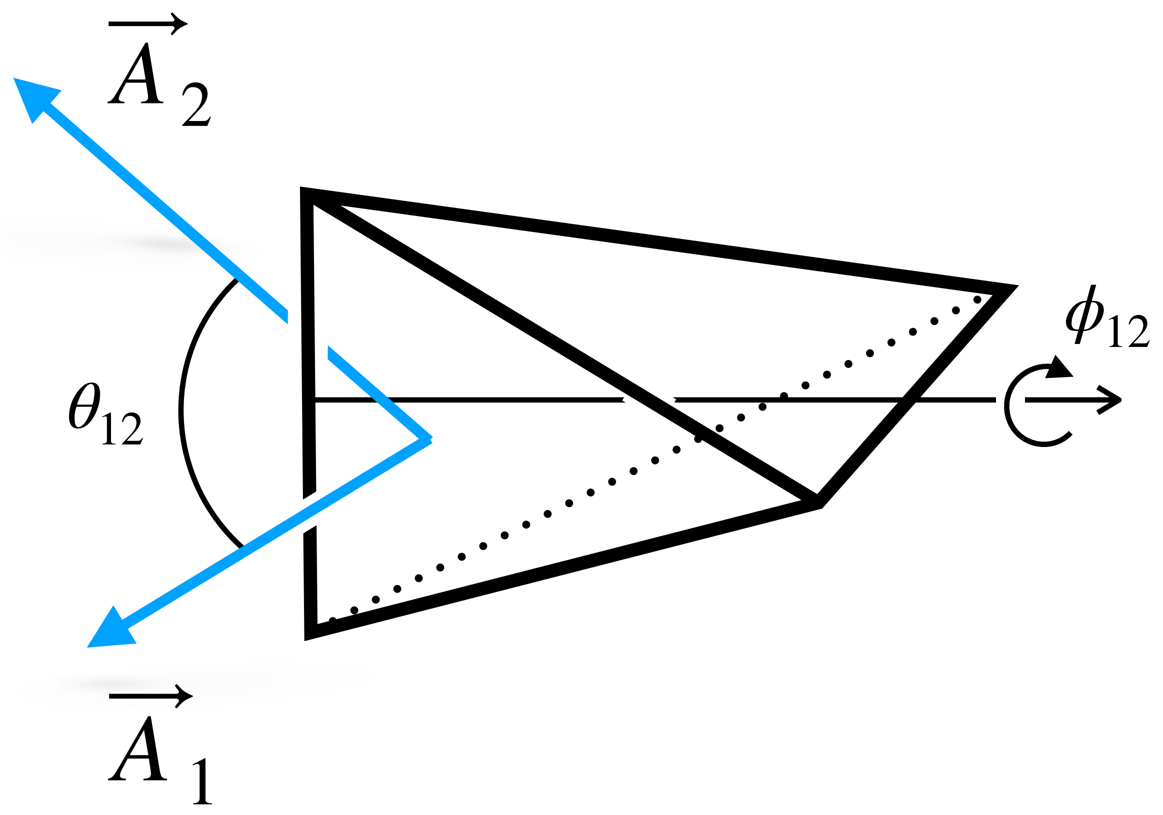
or **a reference system for chirality (but we are studying an isolated tetrahedron)**

quantum equiarea tetrahedron

volume spectrum

phase space of equiarea tetrahedron

canonical variables $\{q, p\} = 1$



$$q = \phi_{12} \qquad \in [-\pi, \pi]$$

$$p = \cos\left(\frac{1}{2}\theta_{12}\right) p_{\max} \qquad \in [0, p_{\max}], \quad p_{\max} = (2j+1)$$

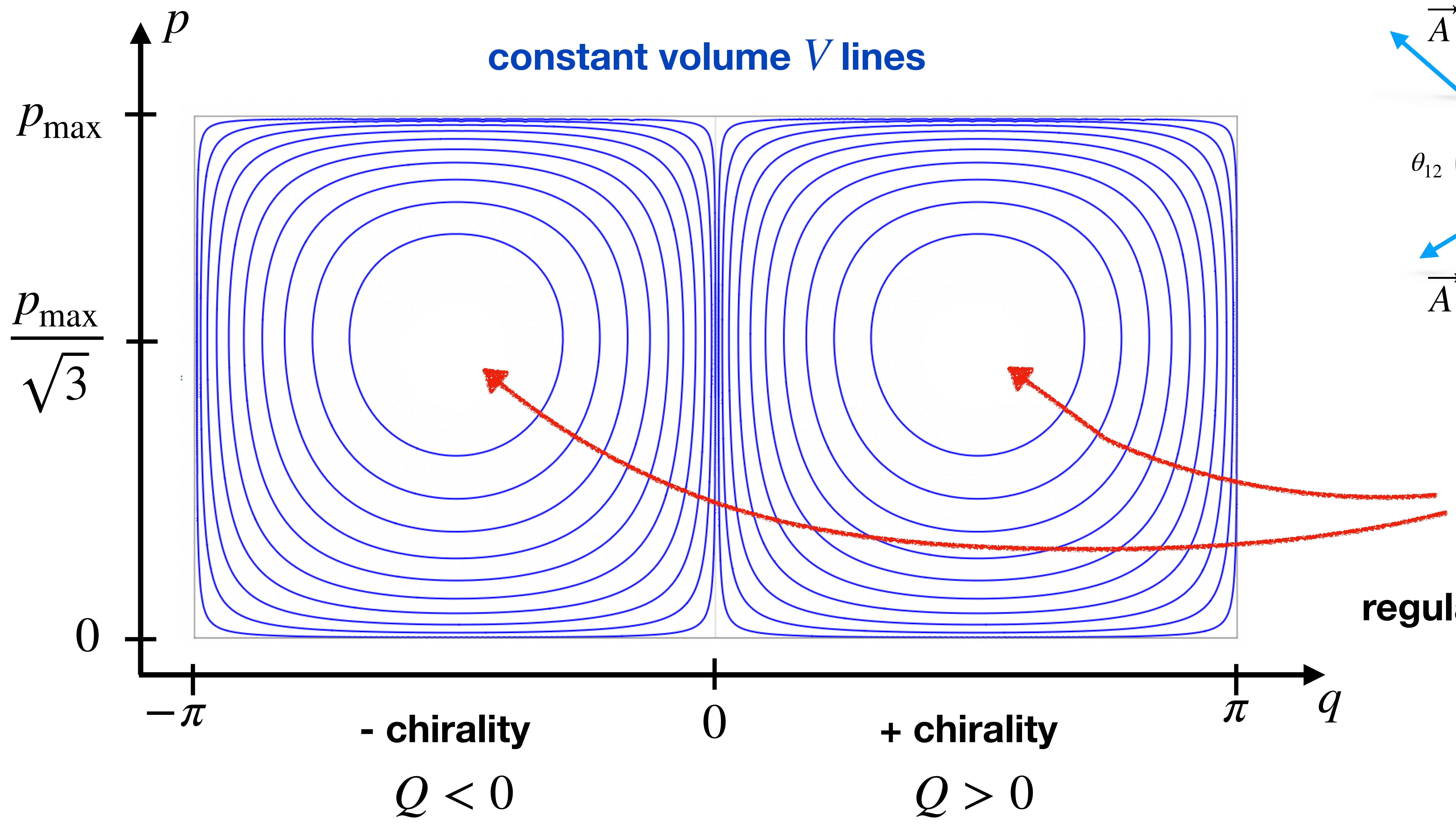
$$V = \frac{2}{3}\sqrt{|\sin q| \frac{p}{p_{\max}}\left(1 - \frac{p^2}{p_{\max}^2}\right)}$$

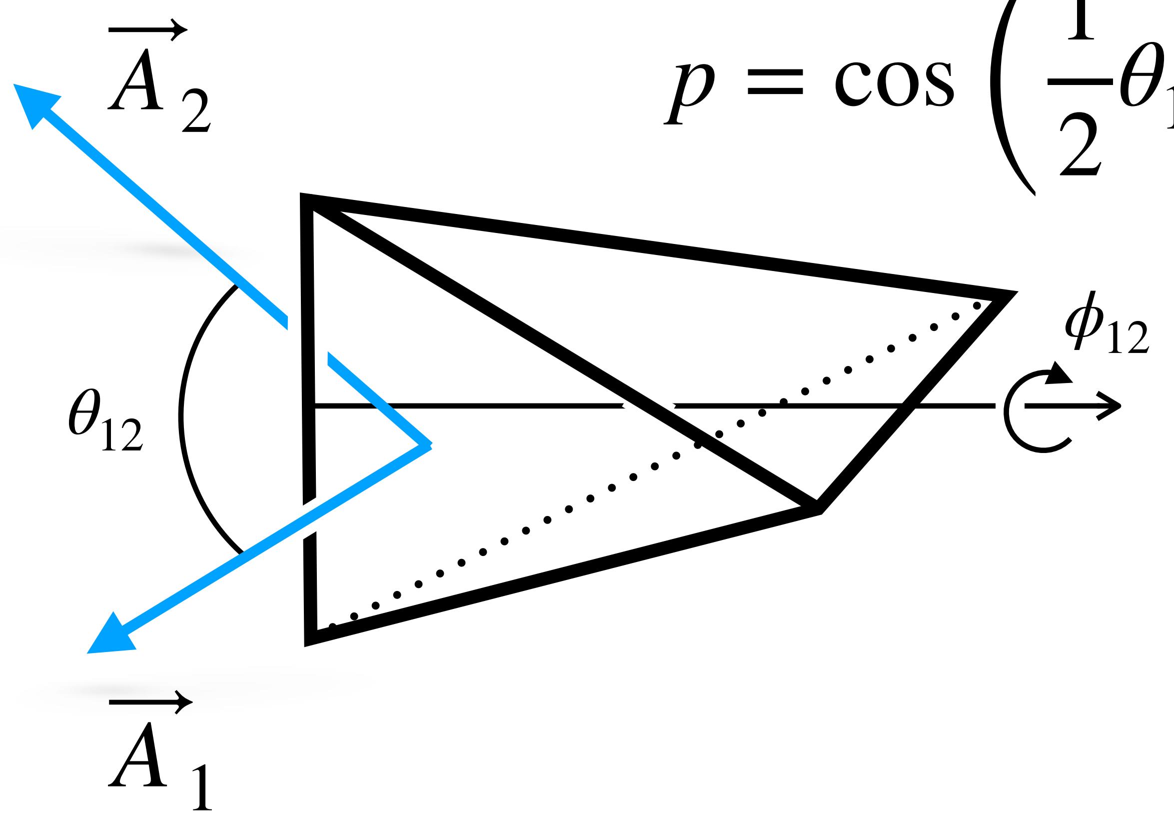
[8] Bianchi and Haggard (2012) Bohr-Sommerfeld quantization of space Phys. Rev. D **86**, 124010 DOI: 10.1103/PhysRevD.86.124010

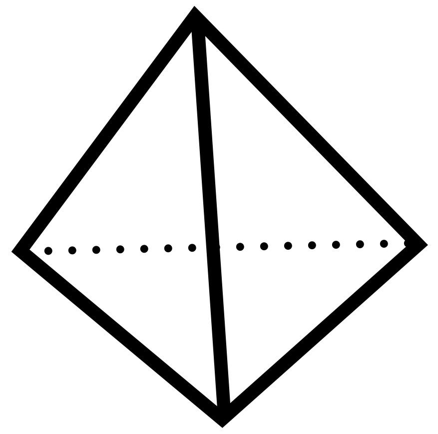
phase space of equiarea tetrahedron

$q = \phi_{12}$

$p = \cos\left(\frac{1}{2}\theta_{12}\right) p_{max}$

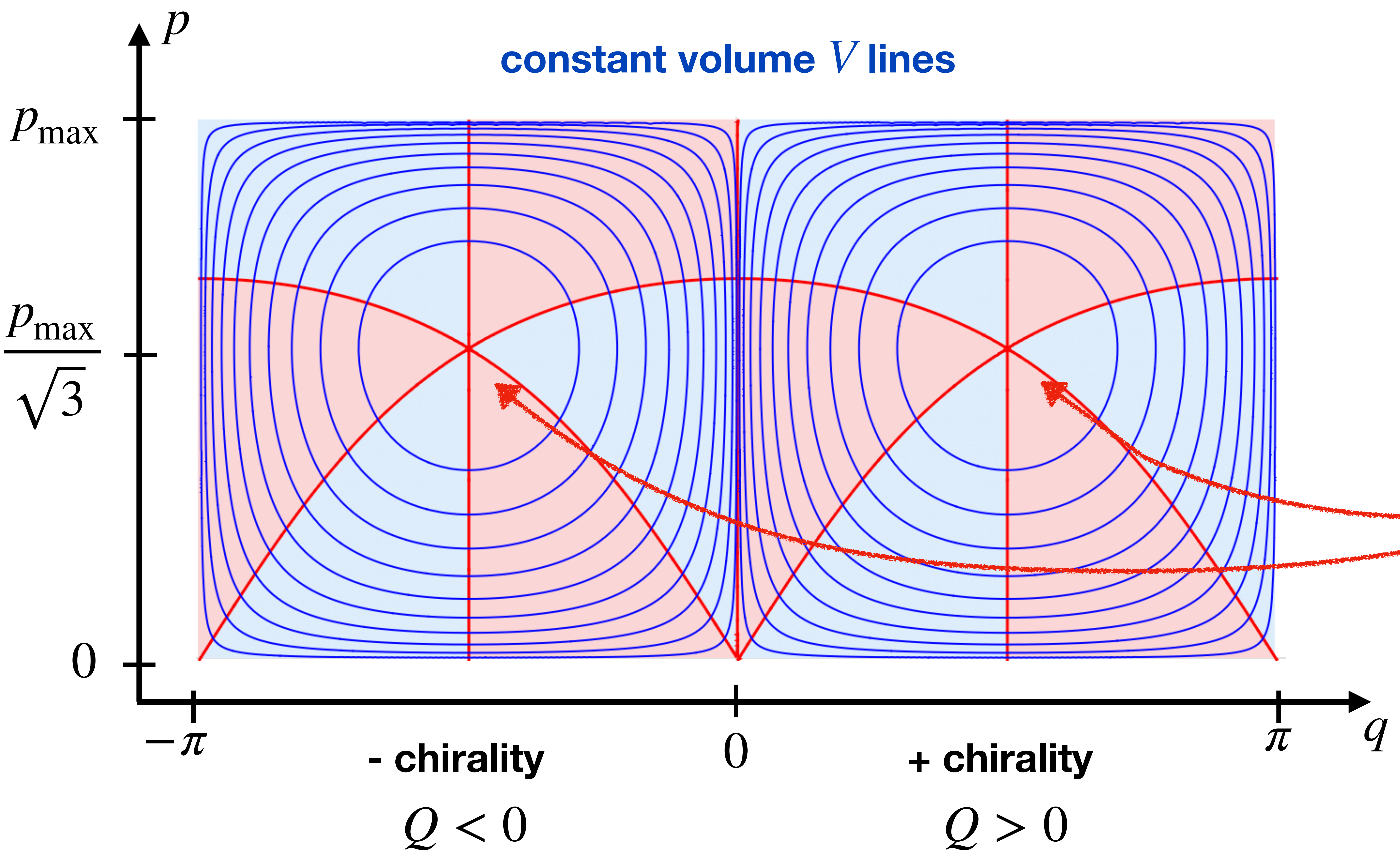






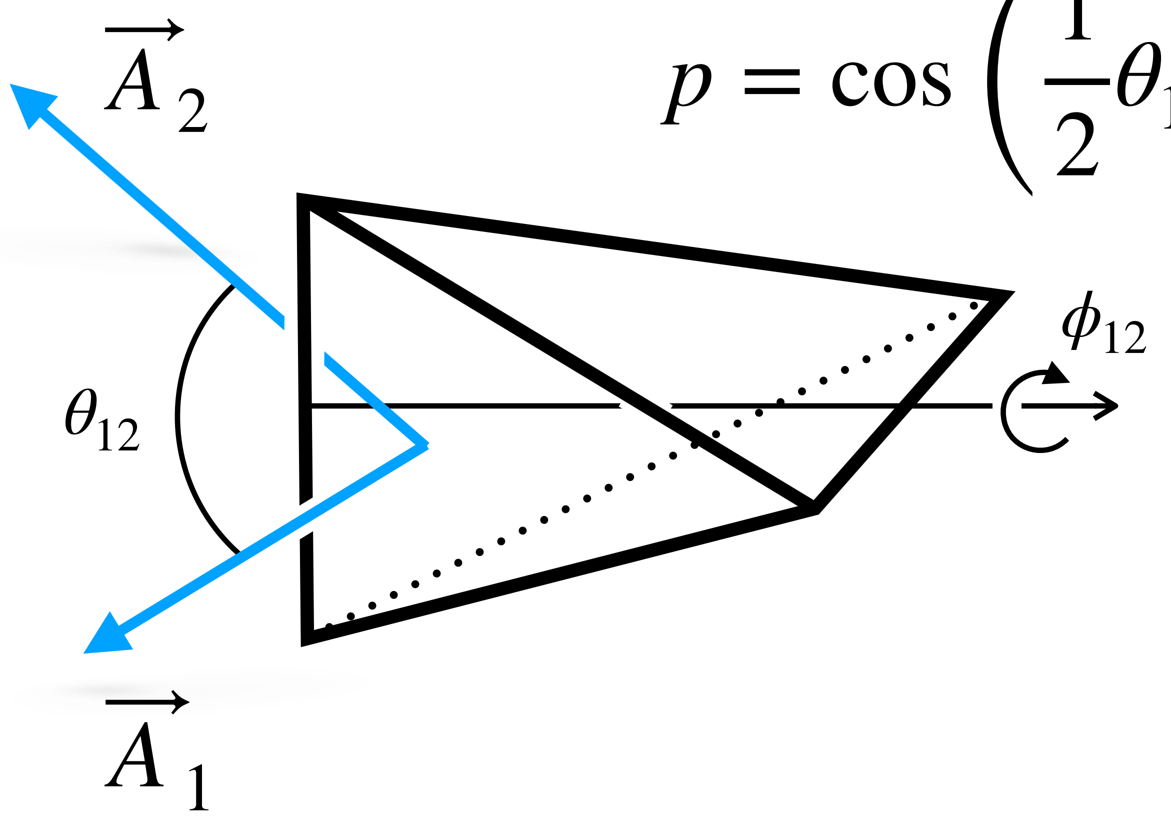
regular tetrahedron

phase space of equiarea tetrahedron

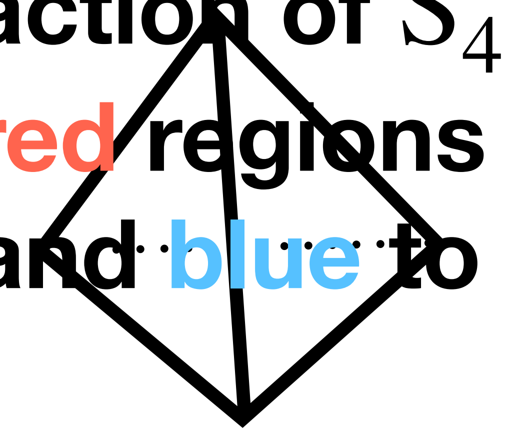


$q = \phi_{12}$

$p = \cos\left(\frac{1}{2}\theta_{12}\right) p_{\max}$

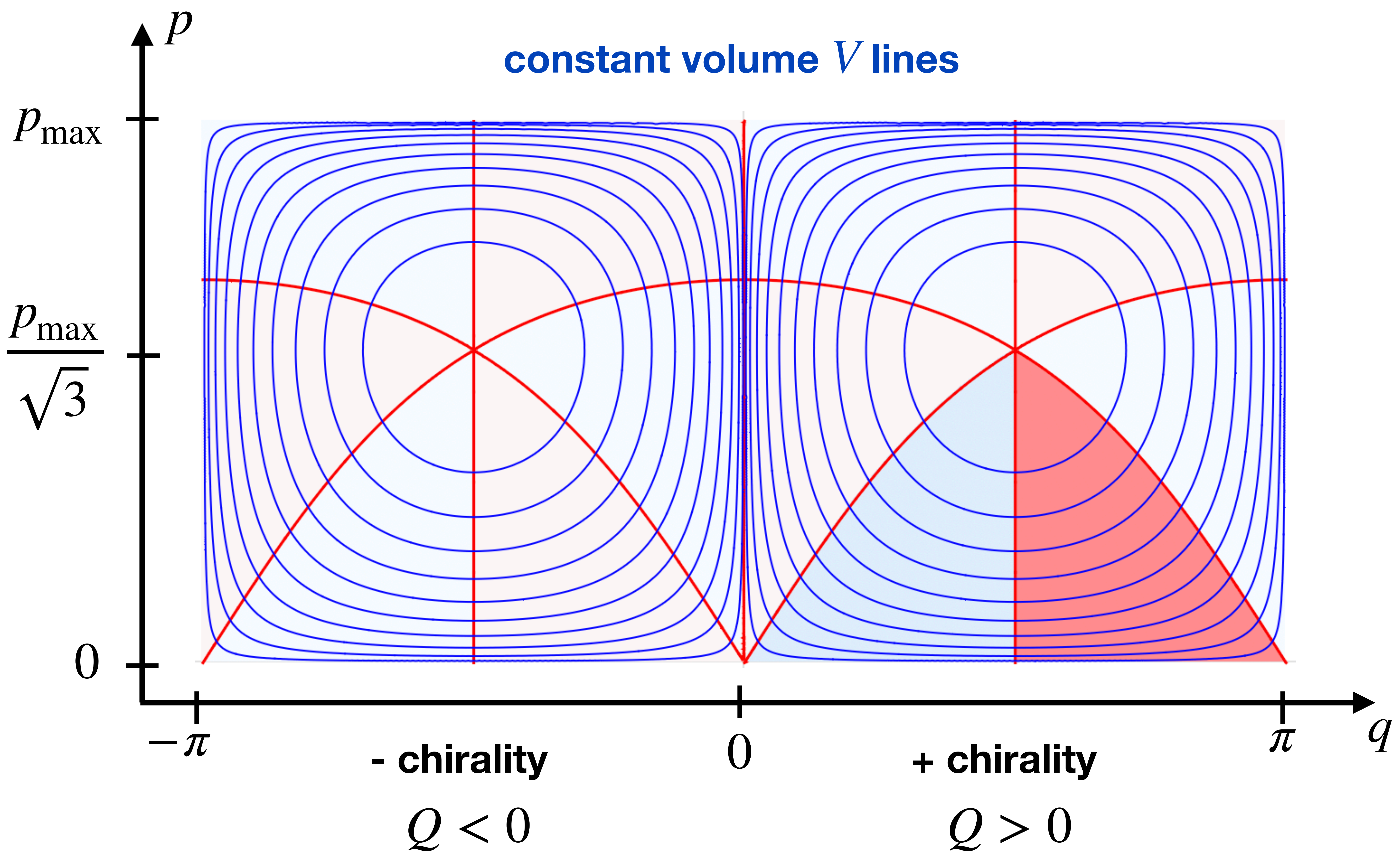


action of S_4 maps
red regions to red regions
and blue to blue.



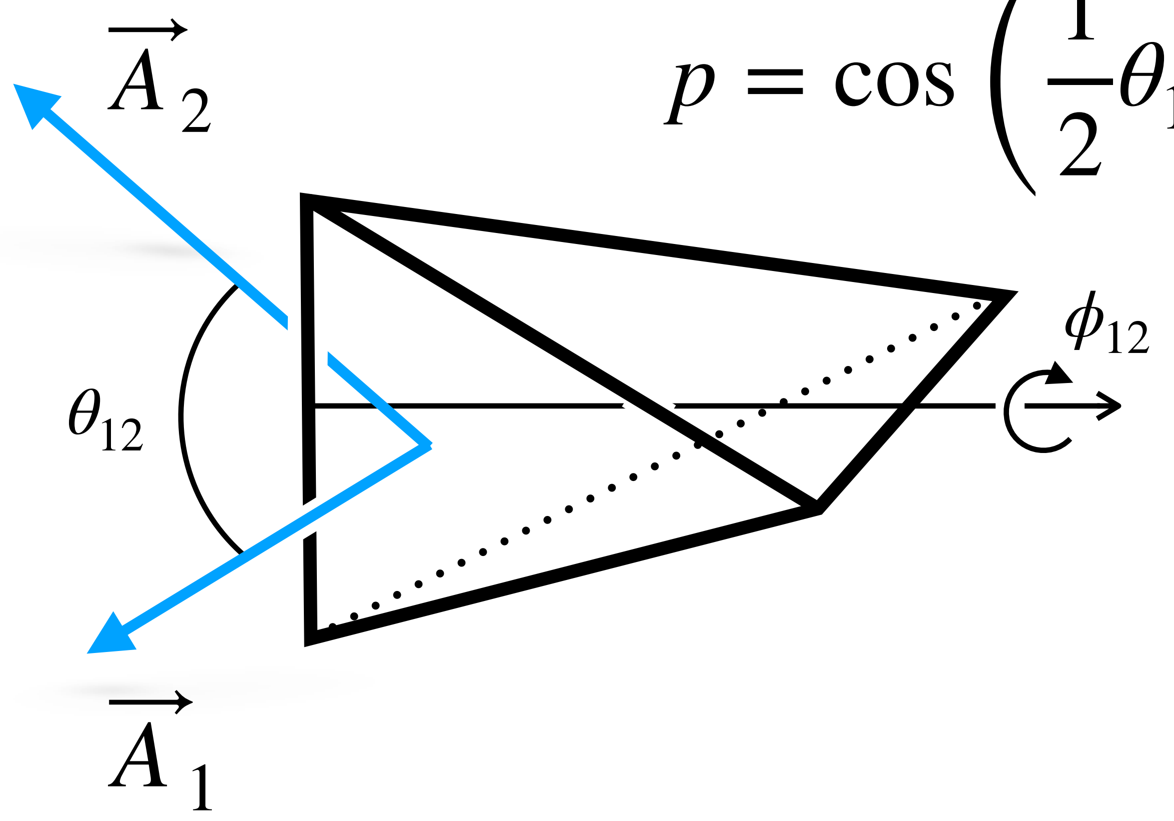
regular tetrahedron

phase space of equiarea tetrahedron



$q = \phi_{12}$

$p = \cos\left(\frac{1}{2}\theta_{12}\right) p_{\max}$

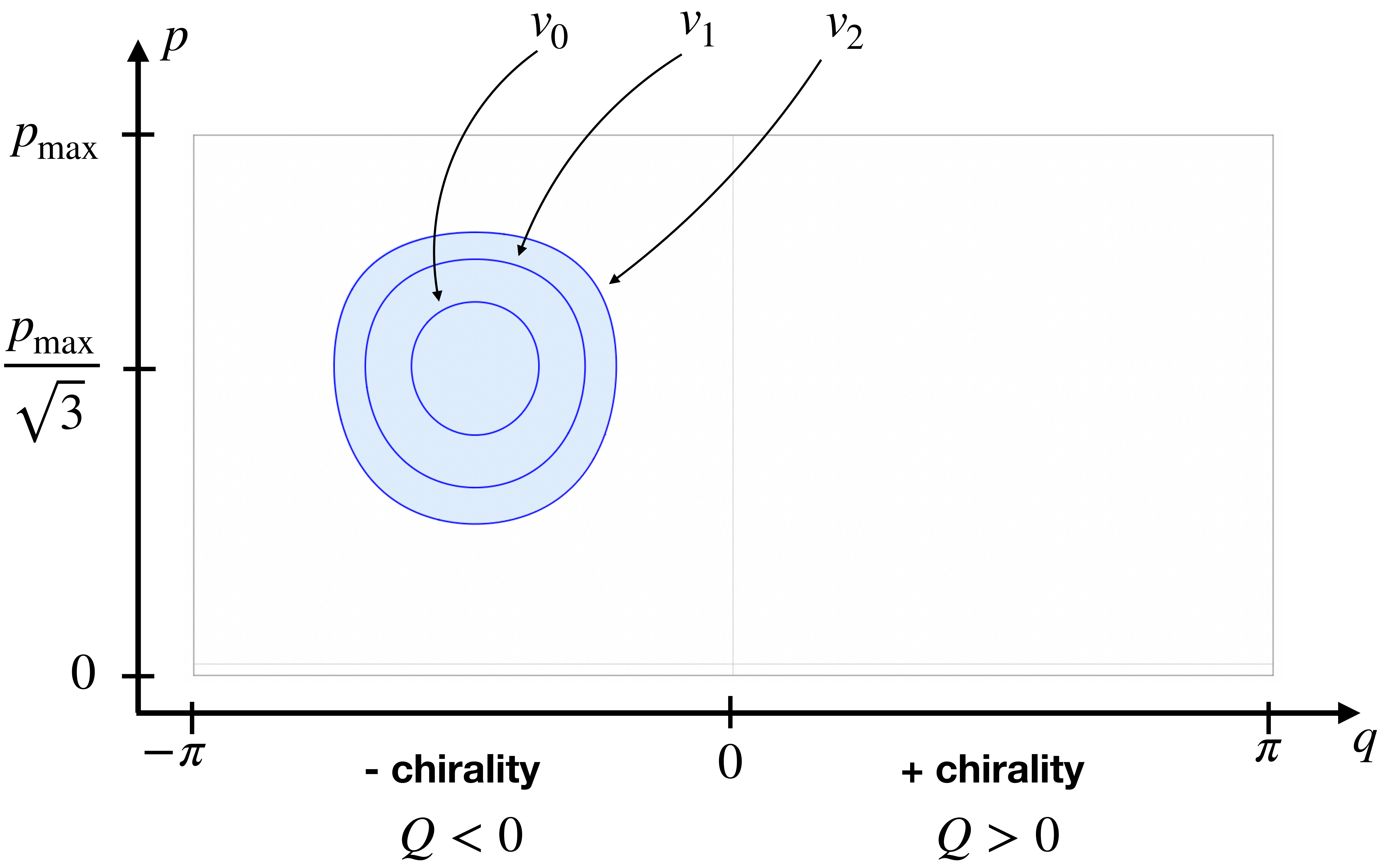


action of S_4 maps red regions to red regions and blue to blue.

identify points in the same orbit

$\Longrightarrow$ resulting space has 1/6 the volume

Bohr-Sommerfeld quantisation

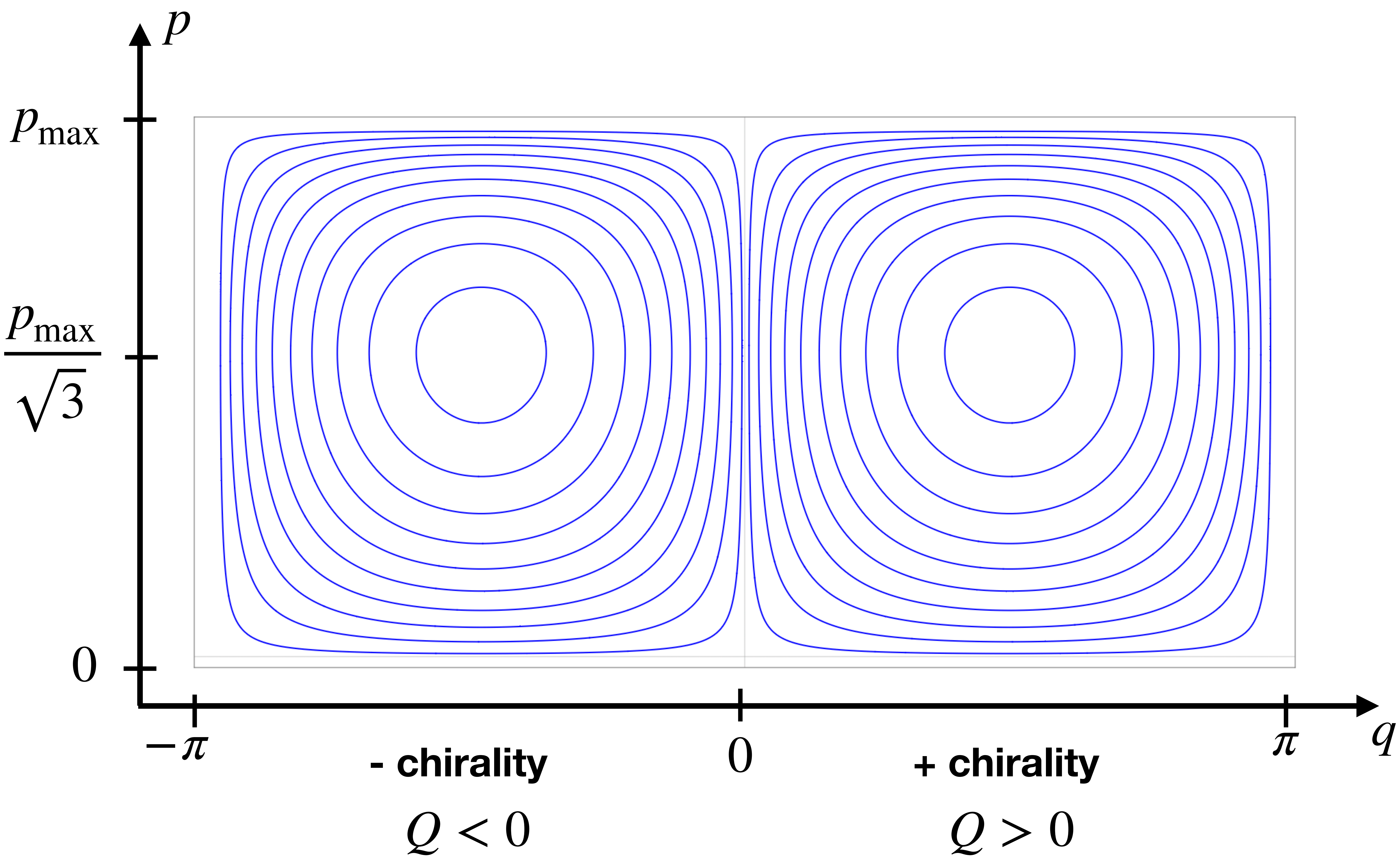


semiclassical approximation

eigenvalues v_n of V satisfy the Bohr-Sommerfeld condition

$$\int_{V \leq v_n} \mathrm{d}p \wedge \mathrm{d}q = 2\pi \left(n + \frac{1}{2} \right)$$

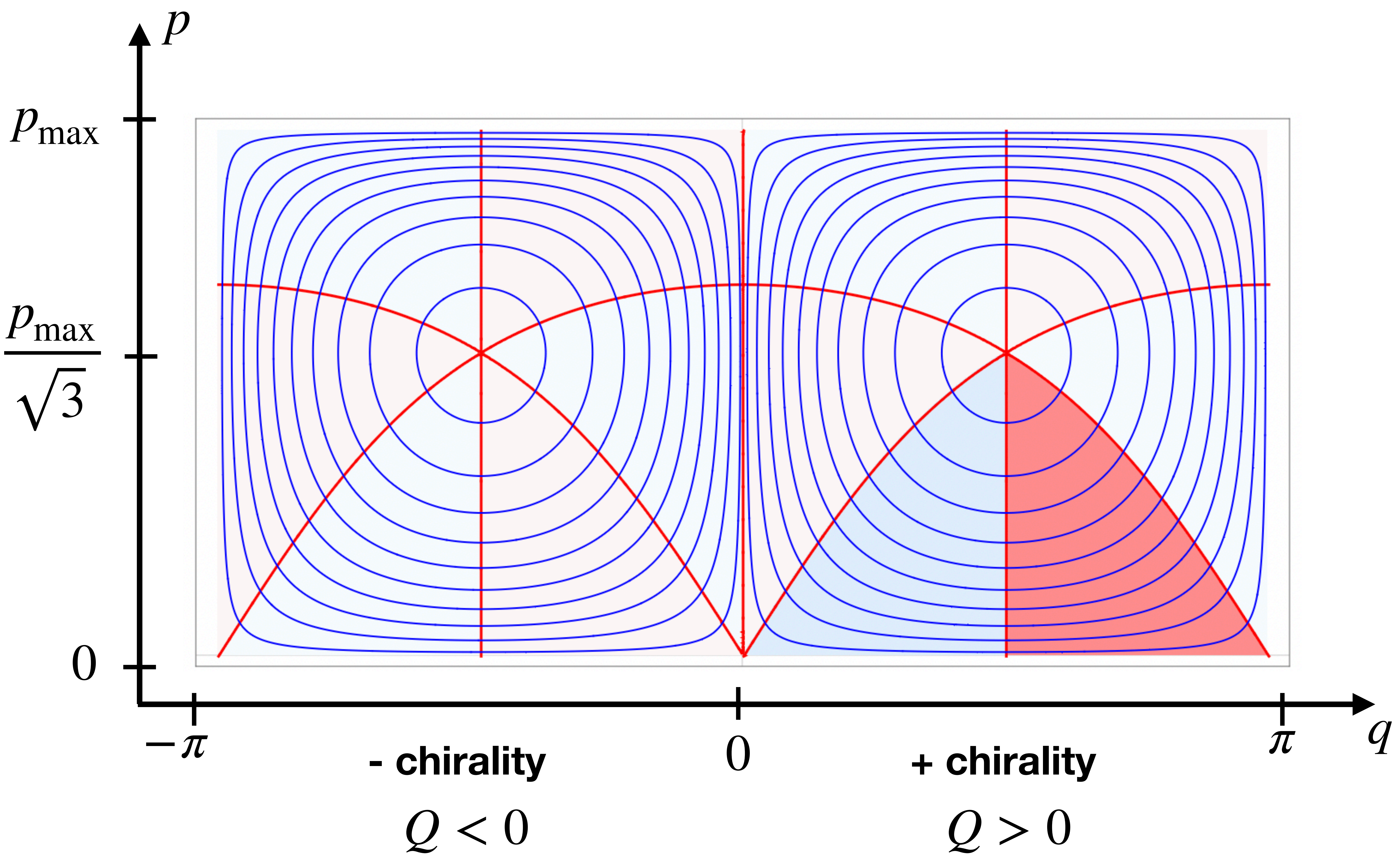
Bohr-Sommerfeld quantisation



Bohr-Sommerfeld condition

$$\int_{V \leq v_n} \mathrm{d}p \wedge \mathrm{d}q = 2\pi \left(n + \frac{1}{2} \right)$$

Bohr-Sommerfeld quantisation

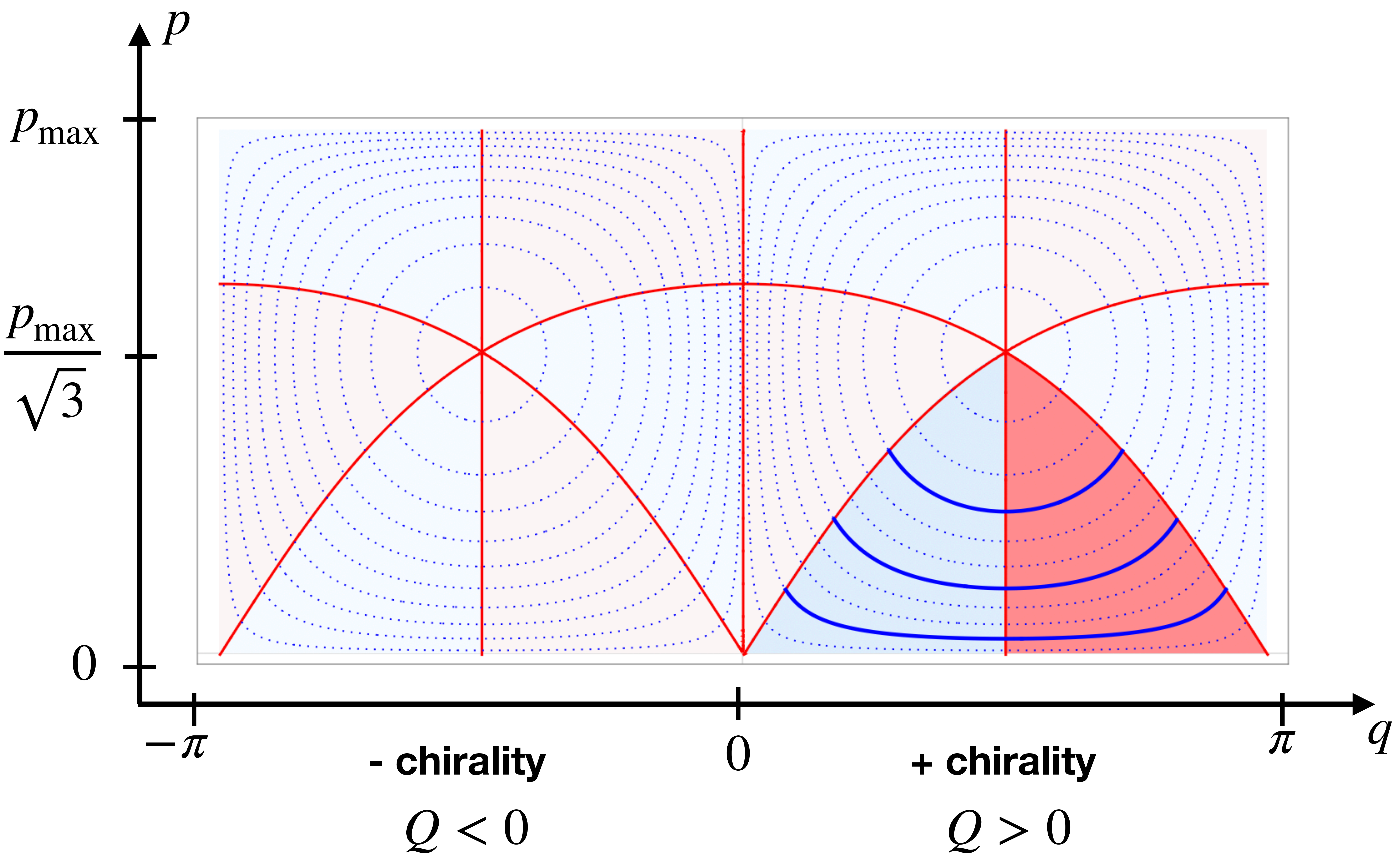


Bohr-Sommerfeld condition

$$\int_{V \leq v_n} \mathrm{d}p \wedge \mathrm{d}q = 2\pi \left(n + \frac{1}{2} \right)$$

after symplectic reduction, only area in the fundamental region counts

Bohr-Sommerfeld quantisation



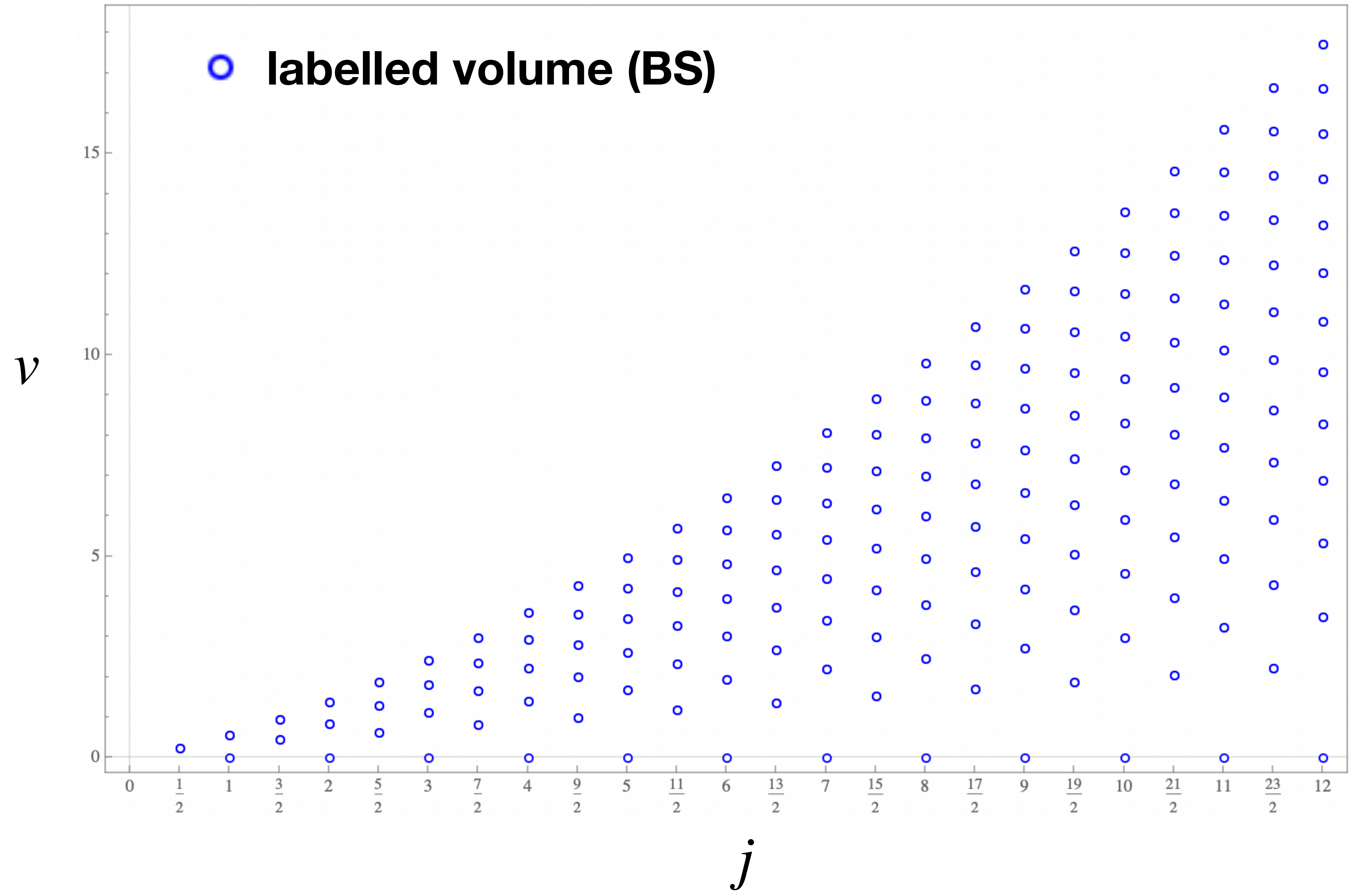
Bohr-Sommerfeld condition

$$\int_{V \le v_n} \mathrm{d}p \wedge \mathrm{d}q = 2\pi \left(n + \frac{1}{2} \right)$$

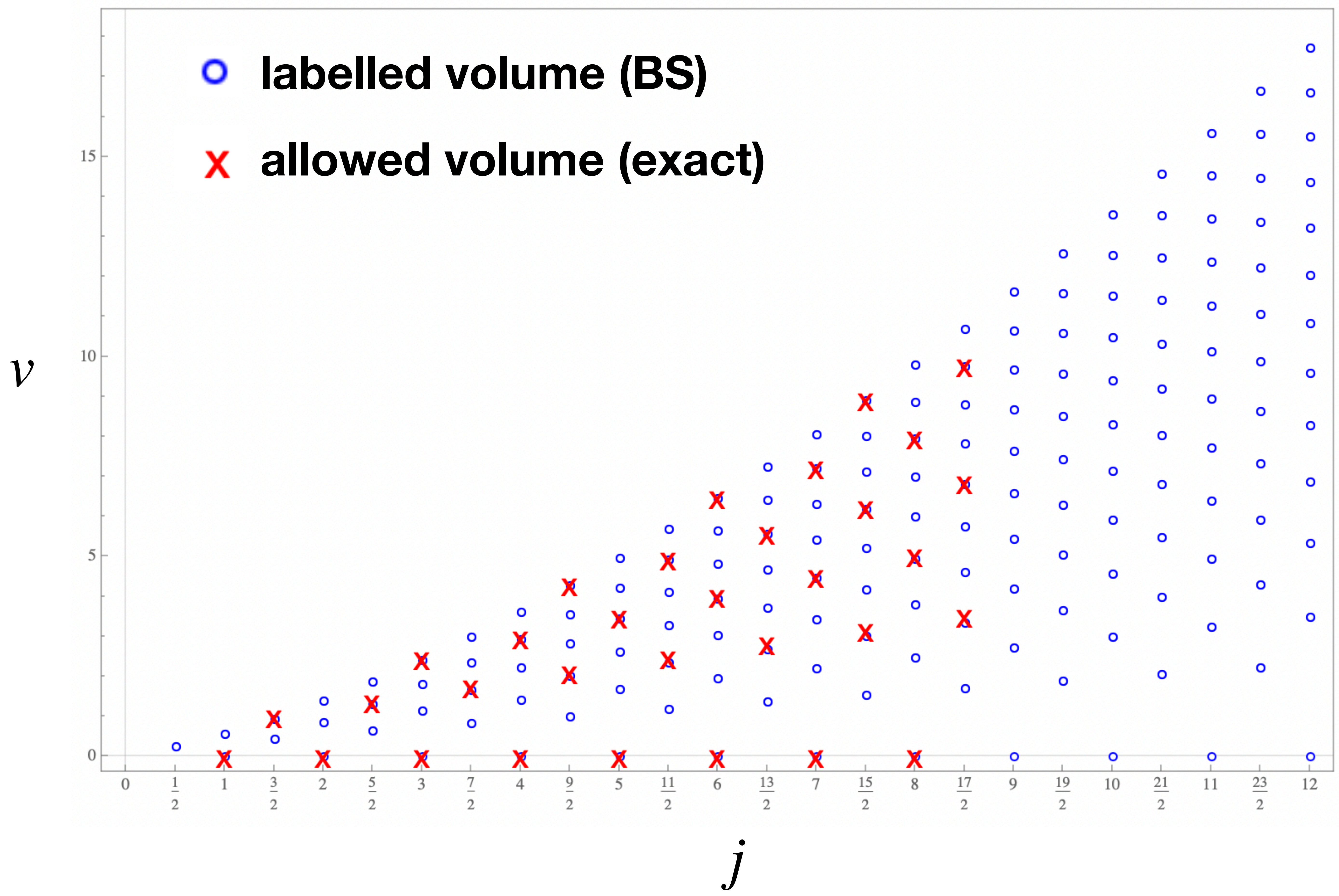
after symplectic reduction, only area in the fundamental region counts

1/6 the area,
1/6 the eigenvalues

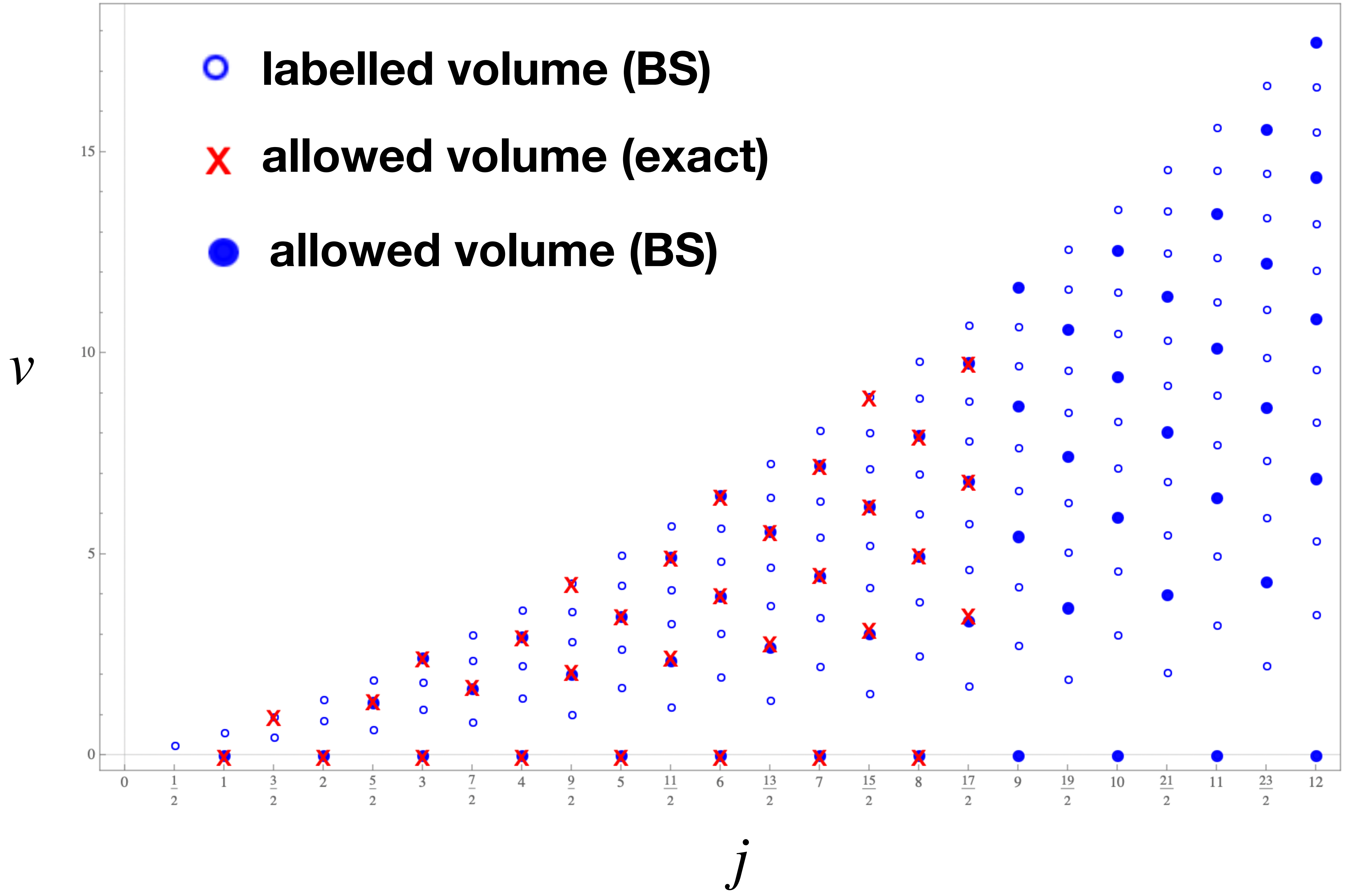
Bohr-Sommerfeld quantisation



Bohr-Sommerfeld quantisation



Bohr-Sommerfeld quantisation



quantum polyhedra

labelled quantum polyhedron

kinematical Hilbert space

$$\mathscr{H}^{\text{kin}} = \bigotimes_{a=1}^{N} \mathscr{H}^{(j_a)}$$

arbitrary number of faces and areas

$\mathrm{SU}(2)$ **action**

$$U(g) = \bigotimes_{a=1}^{N} D^{j_a}(g)$$

physical Hilbert space

$$\mathscr{H}^{(0)} = \mathrm{Inv}_{\mathrm{SU}(2)}\, \mathscr{H}^{\text{kin}}$$

constraint

$$\vec{J} = \sum_{a=1}^{N} \vec{J}_a = 0$$

projector

$$P^{(0)} = \int_{\mathrm{SU}(2)} \mathrm{d}g\, U(g)$$

$$\dim \mathscr{H}^{(0)} = \mathrm{tr}\, P^{(0)} = \int_{\mathrm{SU}(2)} \mathrm{d}g \prod_{a=1}^{N} \mathrm{tr}\, D^{j_a}(g)$$

permutation group

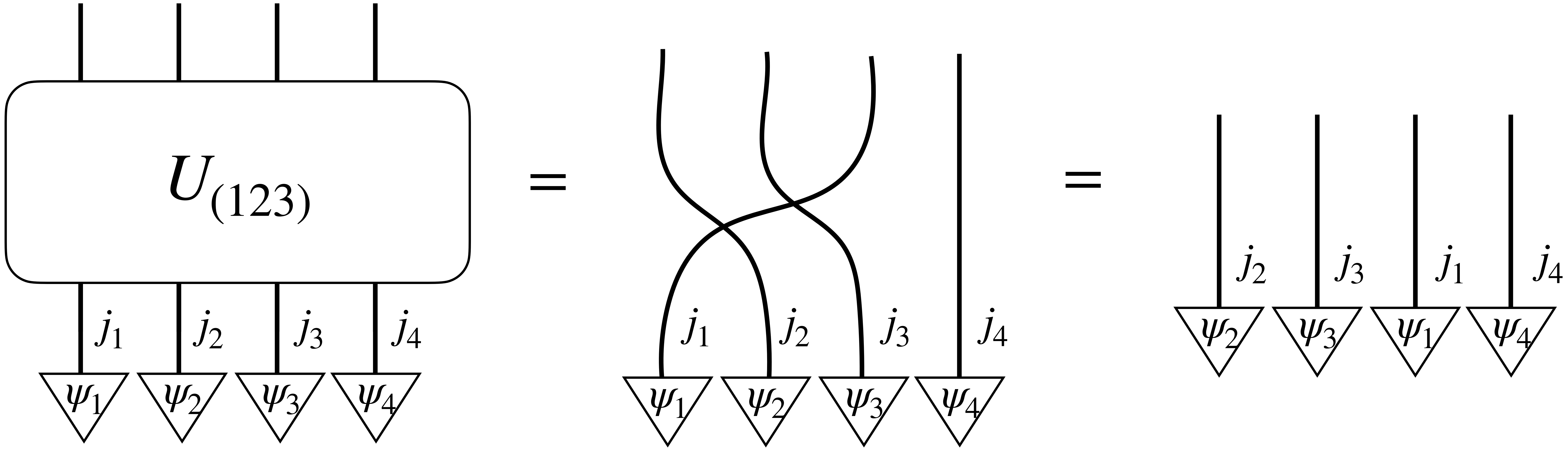
kinematical Hilbert space

$$\mathscr{H}^{\text{kin}} = \bigotimes_{a=1}^{N} \mathscr{H}^{(j_a)}$$

$$U_\sigma : \bigotimes_{a=1}^{N} \mathscr{H}^{(j_a)} \longrightarrow \bigotimes_{a=1}^{N} \mathscr{H}^{(j_{\sigma(a)})}$$

action of the permutation group

$$U_\sigma \bigotimes_{a=1}^{N} |\psi_a\rangle = \bigotimes_{a=1}^{N} |\psi_{\sigma(a)}\rangle$$



quantum polyhedron

kinematical Hilbert space

$$\mathscr{H}^{\text{kin}} = \bigoplus_{\vec{j}\in \text{Perm}(M)} \left(\bigotimes_{a=1}^{N} \mathscr{H}^{(j_a)} \right)$$

M **is the *multiset* of spin labels**
$\text{Perm}(M)$ **is the set of *distinct* permutations of its entries**

$$M = \{1,1,1,2\} \qquad \text{Perm}(M) = \{(1,1,1,2), (1,1,2,1), (1,2,1,1), (2,1,1,1)\}$$

$$\mathscr{H}^{\text{kin}} = \left(\mathscr{H}^{(1)} \otimes \mathscr{H}^{(1)} \otimes \mathscr{H}^{(1)} \otimes \mathscr{H}^{(2)}\right) \oplus \left(\mathscr{H}^{(1)} \otimes \mathscr{H}^{(1)} \otimes \mathscr{H}^{(2)} \otimes \mathscr{H}^{(1)}\right)$$
$$\oplus \left(\mathscr{H}^{(1)} \otimes \mathscr{H}^{(2)} \otimes \mathscr{H}^{(1)} \otimes \mathscr{H}^{(1)}\right) \oplus \left(\mathscr{H}^{(2)} \otimes \mathscr{H}^{(1)} \otimes \mathscr{H}^{(1)} \otimes \mathscr{H}^{(1)}\right)$$

quantum polyhedron

kinematical Hilbert space

$$\mathscr{H}^{\text{kin}}[M] = \bigoplus_{\vec{j}\in\text{Perm}(M)} \left(\bigotimes_{a=1}^{N} \mathscr{H}^{(j_a)} \right)$$

physical Hilbert space

$$\mathscr{H}^{\text{phys}}[M] = \text{Inv}_{\text{SU(2)}\times\text{S}_N}\, \mathscr{H}^{\text{kin}}[M]$$

$$\dim \mathscr{H}^{\text{phys}}[M] = \text{tr}\, P^{(0)} P^{\text{sym}} = \frac{1}{N!} \int_{\text{SU(2)}} \text{d}g \sum_{\sigma\in\text{S}_N} \text{tr}\, U_\sigma U(g)$$

$$\dim \mathscr{H}^{\text{phys}}[M] = \frac{|\,\text{Perm}(M)\,|}{N!} \int_{\text{SU(2)}} \text{d}g \prod_{j\in M} \sum_{\lambda \vdash \mu_M(j)} C_\lambda \prod_{k=1}^{\mu_M(j)} [\text{tr}\, D^j(g^k)]^{\mu_\lambda(k)}$$

arbitrary faces, fixed area

kinematical Hilbert space

$$\mathscr{H}^{\text{kin}}[J] = \bigoplus_{M \in \mathscr{M}_J} \mathscr{H}^{\text{kin}}[M]$$

$\mathscr{M}_J$ **is the *set* of multisets of spin labels such that:** $\sum_a j_a = J$

$$\{1,1,1,1\}, \quad \left\{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\}, \quad \{2,1,1,1,1\} \in \mathscr{M}_4$$

arbitrary faces, fixed area

kinematical Hilbert space

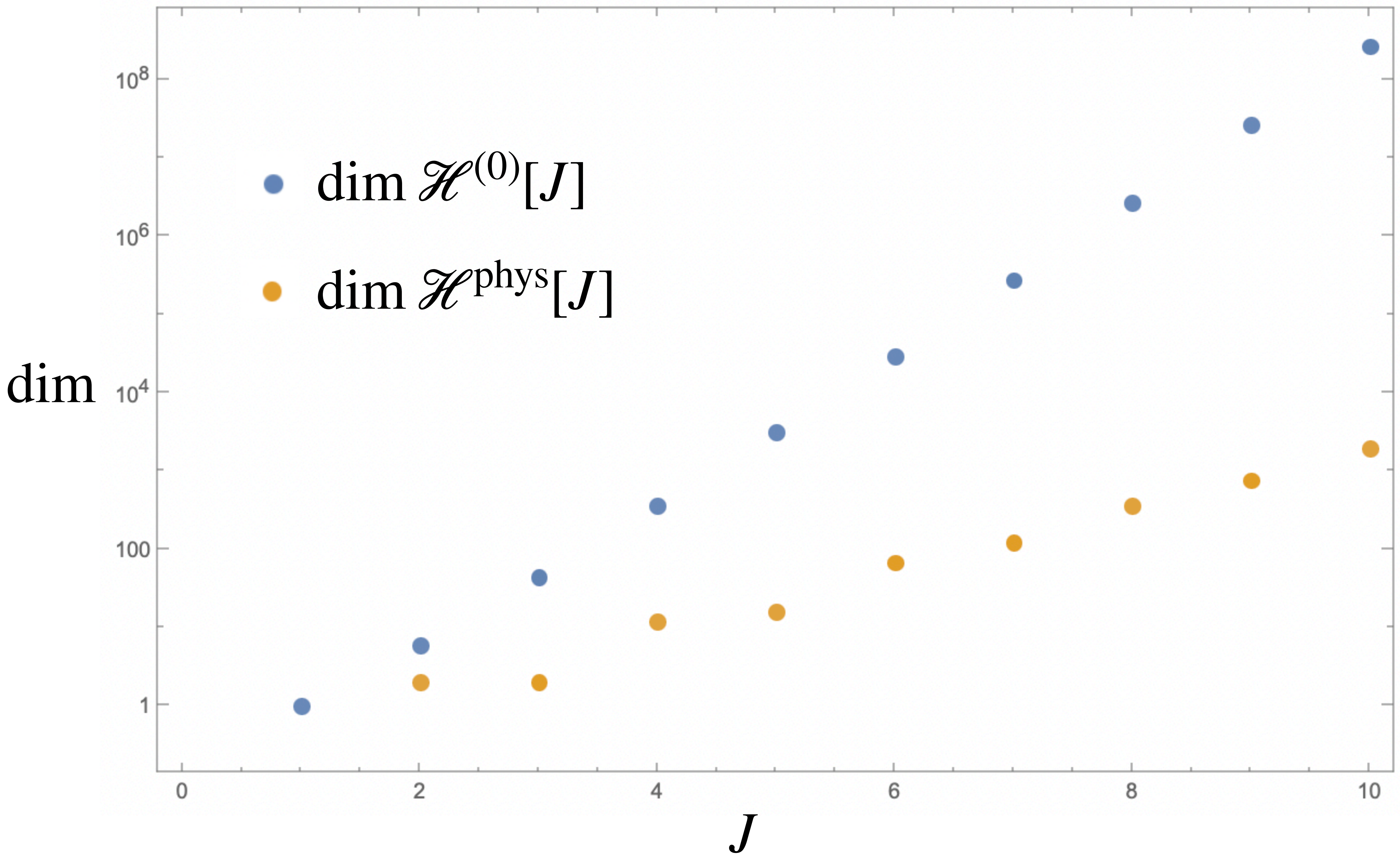
$$\mathscr{H}^{\text{kin}}[J] = \bigoplus_{M \in \mathscr{M}_J} \mathscr{H}^{\text{kin}}[M]$$

$\mathscr{M}_J$ **is the *set* of multisets of spin labels such that:** $\sum_a j_a = J$

$$\dim \mathscr{H}^{(0)}[J] = \sum_{M \in \mathscr{M}_J} \dim \text{Inv}_{\text{SU(2)}} \mathscr{H}^{\text{kin}}[M]$$

$$\dim \mathscr{H}^{\text{phys}}[J] = \sum_{M \in \mathscr{M}_J} \dim \text{Inv}_{\text{SU(2)} \times \text{S}_{|M|}} \mathscr{H}^{\text{kin}}[M]$$

arbitrary faces, fixed area



semiclassical limit ($J \gg 1$) affected

distinct faces

kinematical Hilbert space

$$\mathscr{H}^{\text{kin}} = \bigoplus_{\vec{j} \in \text{Perm}(M)} \left(\bigotimes_{a=1}^{N} \mathscr{H}^{(j_a)} \right)$$

$$j_a \neq j_b$$

$$\dim \mathscr{H}^{\text{phys}} = \underbrace{\frac{|\text{Perm}(M)|}{N!}}_{1} \int_{\text{SU}(2)} \text{d}g \prod_{a=1}^{N} \text{tr}\, D^{j_a}(g)$$

$$\dim \mathscr{H}^{\text{phys}} = \underbrace{\frac{|\text{Perm}(M)|}{N!}}_{1} \int_{\text{SU}(2)} \text{d}g \prod_{j \in M} \underbrace{\sum_{\lambda \vdash \mu_M(j)}}_{1} C_\lambda \prod_{k=1}^{\mu_M(j)} [\text{tr}\, D^j(g^k)]^{\mu_\lambda(k)}$$

distinct faces

kinematical Hilbert space

$$\mathscr{H}^{\text{kin}} = \bigoplus_{\vec{j}\in\text{Perm}(M)} \left(\bigotimes_{a=1}^{N} \mathscr{H}^{(j_a)} \right)$$

$$j_a \neq j_b$$

$$\tilde{\mathscr{H}}^{\text{kin}} = \bigotimes_{a=1}^{N} \mathscr{H}^{(j_a)}$$

$$\dim \text{Inv}_{\text{SU}(2)\times \text{S}_N} \mathscr{H}^{\text{kin}} = \int_{\text{SU}(2)} \text{d}g \prod_{a=1}^{N} \text{tr}\, D^{j_a}(g) = \dim \text{Inv}_{\text{SU}(2)} \tilde{\mathscr{H}}^{\text{kin}}$$

same dimension as the labelled polyhedron!

physical labels

***in*distinguishable particles in bounding potentials**

same energy level

$$0 = |\varepsilon\uparrow\rangle|\varepsilon\uparrow\rangle$$

$$|\psi\rangle = |\varepsilon\uparrow\rangle|\varepsilon\downarrow\rangle$$

$$|\psi\rangle = |\varepsilon\downarrow\rangle|\varepsilon\uparrow\rangle$$

$$0 = |\varepsilon\downarrow\rangle|\varepsilon\downarrow\rangle$$

different energy levels

$$|\varepsilon\uparrow\rangle|\tilde{\varepsilon}\uparrow\rangle$$

$$|\varepsilon\uparrow\rangle|\tilde{\varepsilon}\downarrow\rangle$$

$$|\varepsilon\downarrow\rangle|\tilde{\varepsilon}\uparrow\rangle$$

$$|\varepsilon\downarrow\rangle|\tilde{\varepsilon}\downarrow\rangle$$

"the spin of the electron at energy ϵ"
is a permutation invariant observable

distinct faces

kinematical Hilbert space

$$\mathscr{H}^{\text{kin}} = \bigoplus_{\vec{j}\in\text{Perm}(M)} \left(\bigotimes_{a=1}^{N} \mathscr{H}^{(j_a)} \right)$$

$$j_a \neq j_b$$

$$\tilde{\mathscr{H}}^{\text{kin}} = \bigotimes_{a=1}^{N} \mathscr{H}^{(j_a)}$$

$$\dim \text{Inv}_{\text{SU(2)}\times \text{S}_N} \mathscr{H}^{\text{kin}} = \int_{\text{SU(2)}} \text{d}g \prod_{a=1}^{N} \text{tr}\, D^{j_a}(g) = \dim \text{Inv}_{\text{SU(2)}} \tilde{\mathscr{H}}^{\text{kin}}$$

"the face with area j_a"
is a permutation invariant label

physical labels allow for more geometries

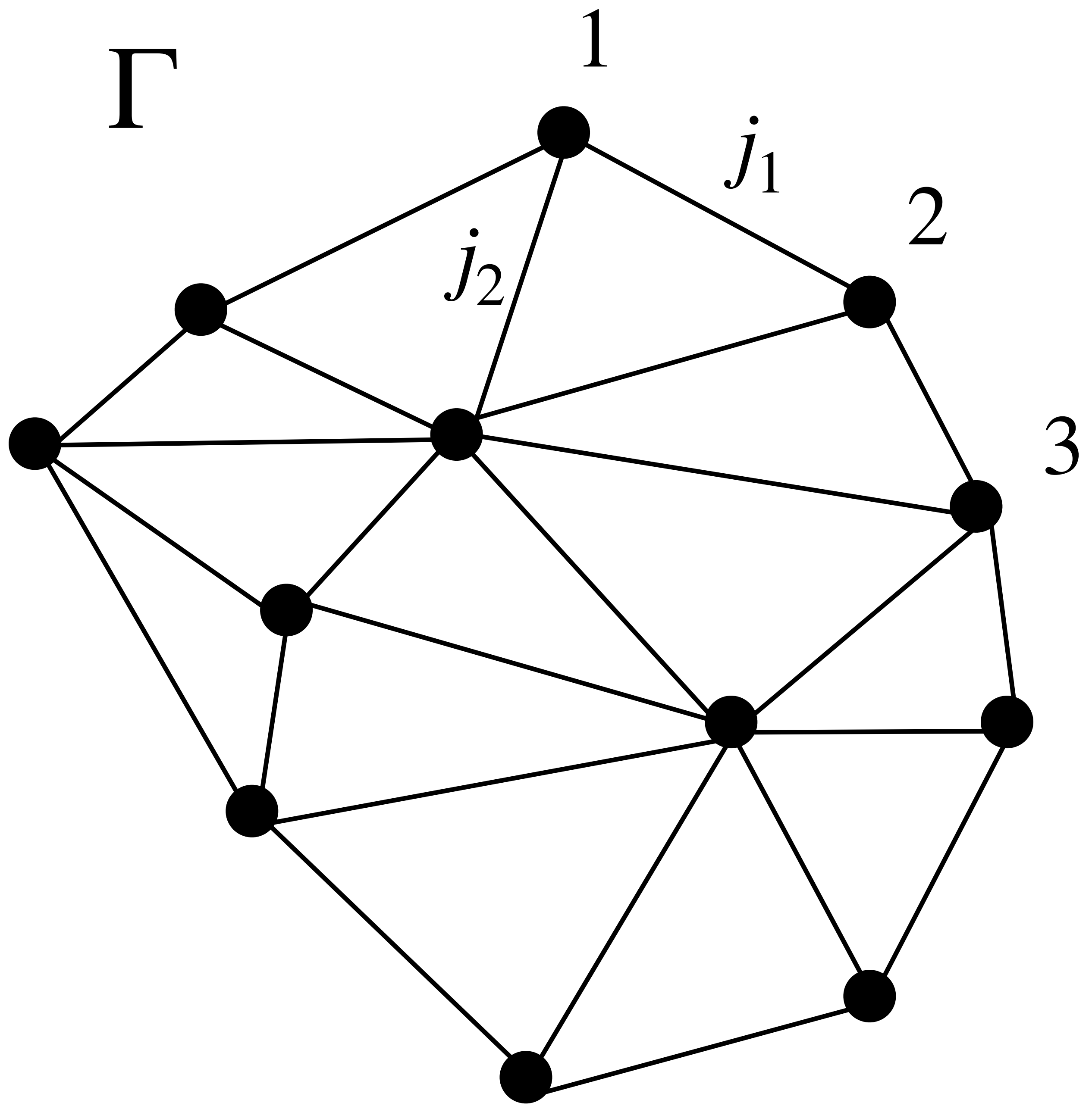
outlook

renaming invariance in LQG

$$\mathscr{H}^{\mathrm{LQG}} = \bigoplus_{\Gamma} \mathscr{H}^{\Gamma}$$

$$\mathscr{H}^{\Gamma} = \bigoplus_{j_1 \cdots j_v} \bigotimes n = 1^{N} \ \mathscr{H}_n[\vec{j}\,]$$

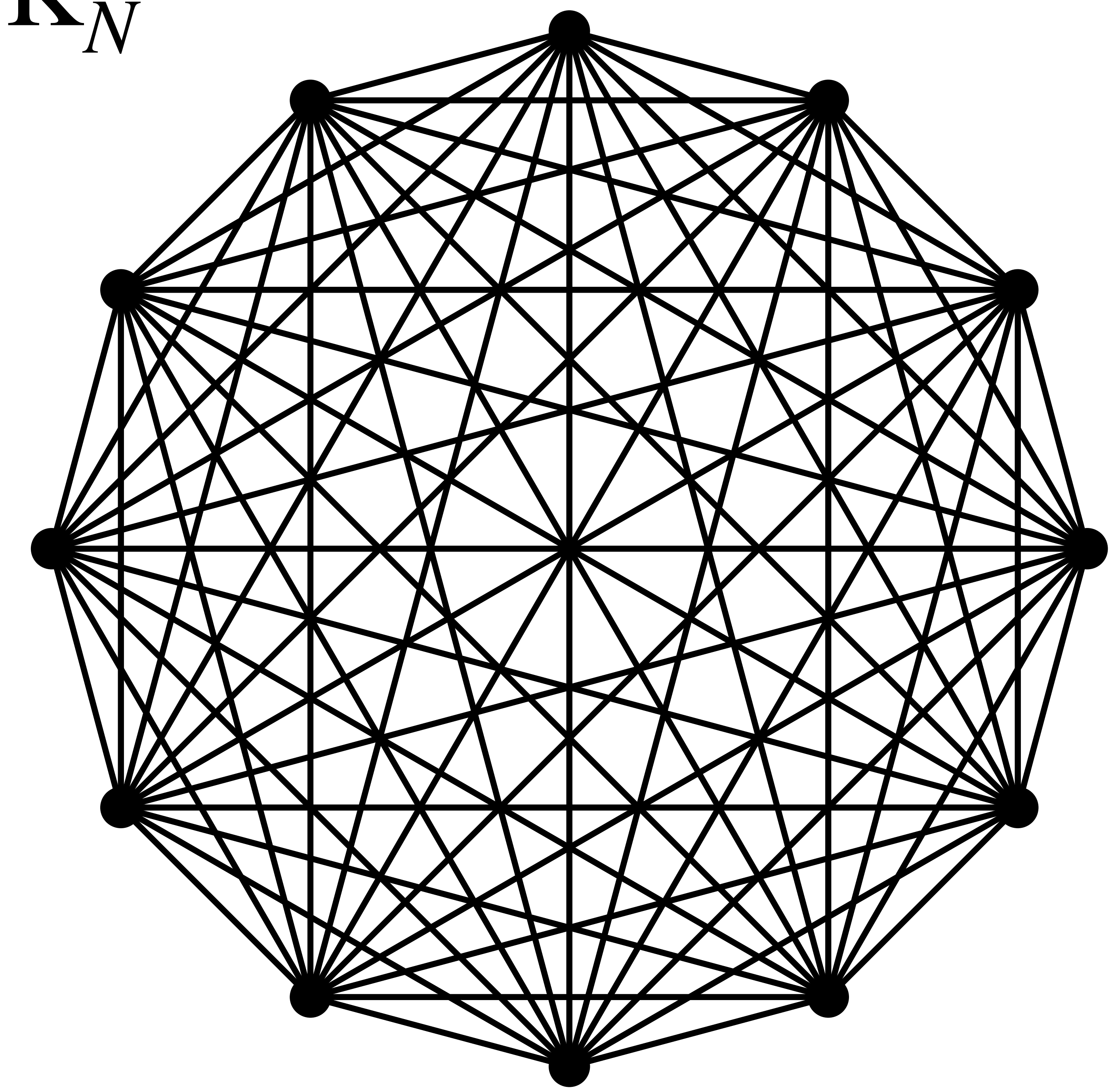
definition relies on labelling of the nodes and edges!



renaming invariance in LQG

$$\mathscr{H}^{\mathrm{LQG}} = \bigoplus_{N} \mathrm{Inv}_{\mathrm{S}_N} \mathscr{H}^{\mathrm{K}_N}$$

K_N



all graphs as embeddings in
the fully connected graph

specified in
permutation invariant way

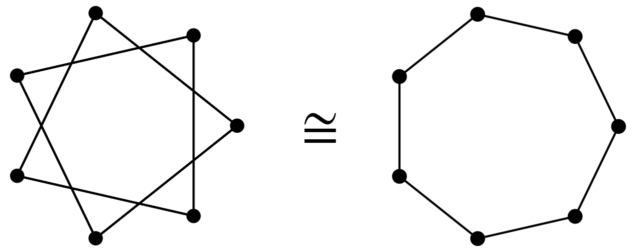
QG exclusion principle in LQG?

quantum polyhedron

observables

	$\vec{J}_a$	$\vec{J}_a \cdot \vec{J}_b$	$\sum_{a,b=1}^{4} \vec{J}_a \cdot \vec{J}_b$
kinematical Hilbert space $\mathscr{H}^{\text{kin}} = \mathscr{H}^{(j)} \otimes \mathscr{H}^{(j)} \otimes \mathscr{H}^{(j)} \otimes \mathscr{H}^{(j)}$	✓	✓	✓
labelled tetrahedron $\mathscr{H}^{(0)} = \text{Inv}_{\text{SU(2)}}\, \mathscr{H}^{\text{kin}}$	✗	✓	✓
unlabelled tetrahedron $\mathscr{H}^{\text{phys}} = \text{Inv}_{\text{SU(2)}\times \text{S}_4}\, \mathscr{H}^{\text{kin}}$	✗	✗	✓

observables are *globalocal*



$$\sum_{a,b=1}^{4} \vec{J}_a \cdot \vec{J}_b$$

kinematical Hilbert space

$$\mathscr{H}^{\text{kin}} = \mathscr{H}^{(j)} \otimes \mathscr{H}^{(j)} \otimes \mathscr{H}^{(j)} \otimes \mathscr{H}^{(j)}$$ ✓

labelled tetrahedron

$$\mathscr{H}^{(0)} = \text{Inv}_{\text{SU(2)}}\, \mathscr{H}^{\text{kin}}$$ ✓

unlabelled tetrahedron

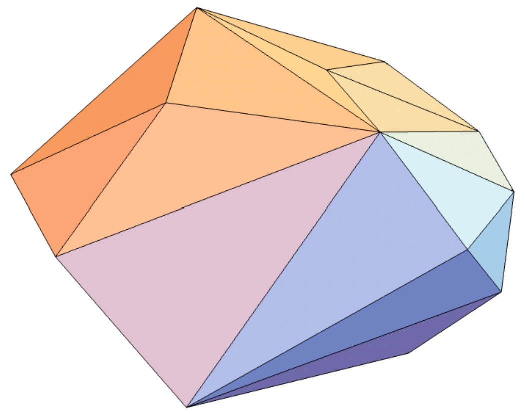
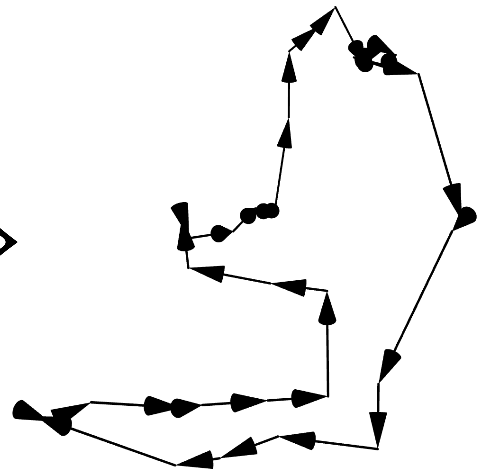
$$\mathscr{H}^{\text{phys}} = \text{Inv}_{\text{SU(2)}\times \text{S}_4}\, \mathscr{H}^{\text{kin}}$$ ✓

Emil Broukal

Eugenio Bianchi

Marios Christodoulou

summary

 $\nLeftrightarrow$ 

$$\dim \mathscr{H}^{\text{phys}}[M] = \frac{|\,\text{Perm}(M)\,|}{N!}\int_{\text{SU}(2)} \mathrm{d}g \prod_{j\in M}\sum_{\lambda\vdash\mu_M(j)} C_\lambda \prod_{k=1}^{\mu_M(j)} [\text{tr}\, D^j(g^k)]^{\mu_\lambda(k)}$$

polyhedra vs polygons

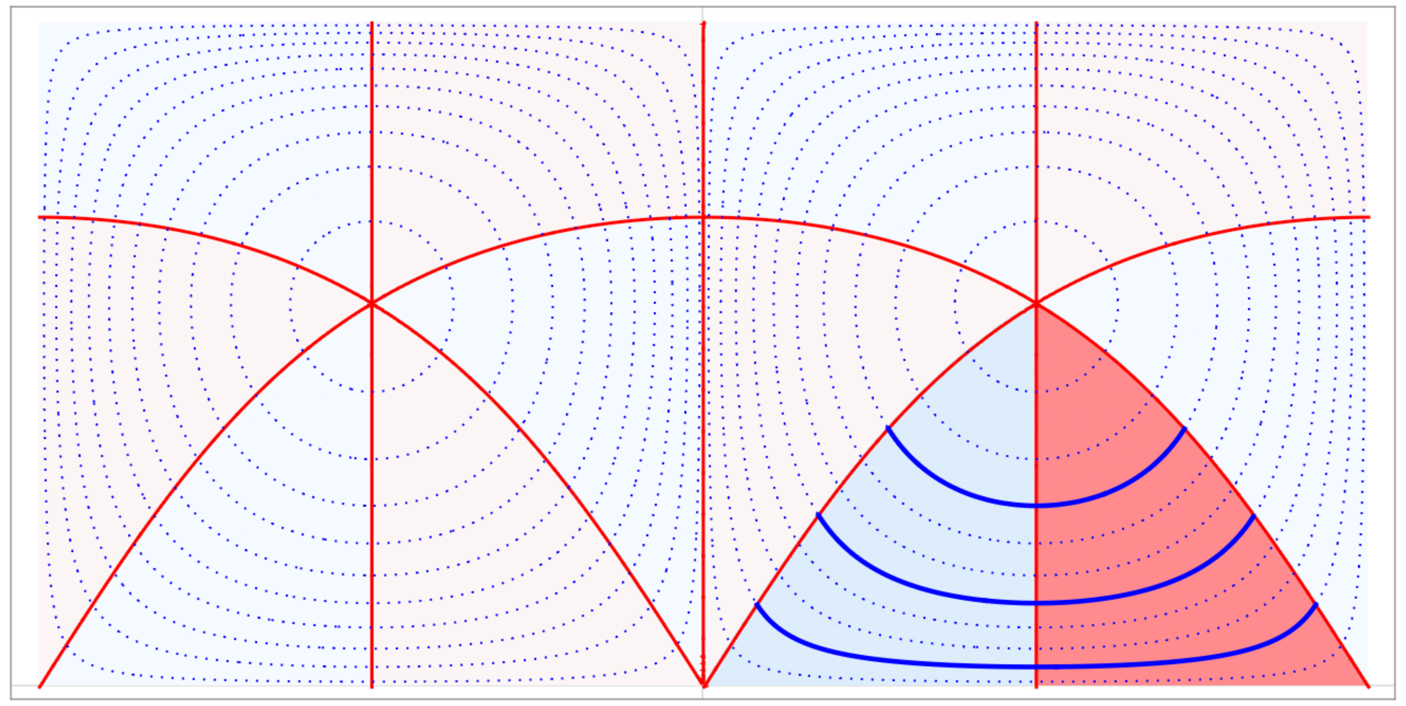
removing unphysical labels $\Longrightarrow$ relabelling (permutation) invariance

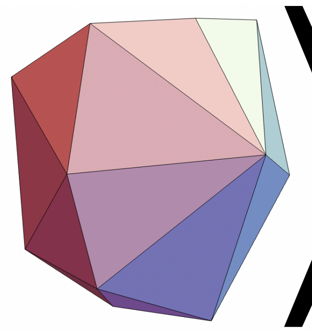
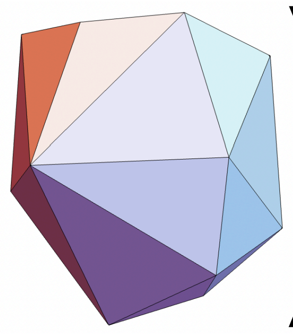
physical effect on quantum geometries: *QG exclusion principle*

different scaling of the dimension

permutation-symmetric polyhedra have no definite chirality

physical labels allow for more geometries



$$|\,\nu^{\text{phys}}\rangle = \Big|$$  $$\Big\rangle + \Big|$$  $$\Big\rangle$$