Spacetime from Causal Structure
MSci Project Report

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Abstract

This is a collection of work done during the academic year 2012-2013 for the MSci Project. We first present a proof that the causal structure of Minkowski spacetime can be used to determine the metric up to an overall conformal factor. This a special case of the result that Hawking *et al.* and Malament [1, 2] proved for general spacetimes. The proof for flat spacetime involves simpler mathematics and has been great help in developing spacetime intuition. Secondly, following a preprint by Bombelli *et al.* [9], we introduce the set of Causal Measure Spaces, which contains both Lorentzian manifolds and Causal Sets and a closeness function that is a real distance function when restricted to compact, distinguishing spaces. This is interesting when wanting to have a quantitative way of telling if a large causet is manifoldlike. We explore and comment on the behaviour of this new function for several examples. A brief summary of our numerical work on causets extracted from manifolds is included as an appendix.

Contents

Introduction

One of the key insights of Special Relativity is that the existence of a "universal speed limit" implies that pairs of events in spacetime do not necessarily influence each other. From this, relativity of simultaneity ensues. Points that do not need to happen in a definite order, because they have no causal connection, appear in different orders to observers depending on their state of motion.

This is covariantly expressed by saying that the events are partially ordered: some pairs of points are commensurable, and some are not. This is what we call the causal structure. There exists a proven result [1, 2], which states that the causal structure encodes almost all of the metric: everything up to a local volume element[1]. This shows a very intimate connection between the global network of cause and effect and the geometry of the universe. The Causal Set approach to quantum gravity [3, 4, 5] seeks to replace the geometry of the Lorentzian manifolds with discrete partially ordered sets called causal sets - causets in short.

The discrete nature of the causet responds to the general idea that spacetime is indeed discrete at a fundamental level. Hints of discreteness are the ultraviolet divergences of the QFTs, the prediction of singularities by General Relativity and the laws of Black Hole Thermodynamics [6, 7, 8]. There are various approaches to the discretisation, but not all are covariant in nature, and will predict deviations from Lorentz invariance. In contrast, the causal set approach relies on the fundamentally covariant causal structure. The causet is the kinematical and dynamical object of study.

During our project time, we were free to explore topics in Causal Set theory. We started by gaining intuition about the spacetime view of Physics and the deep connection between causal structure and geometry by producing a proof by construction of the result of Hawking *et al.* and Malament in the special case of 3+1 Minkowski spacetime. This is presented in Section 1.

An important question in Causal Set theory is how to measure how closely a given causet approaches a manifold. There exists the framework of Causal Measure Spaces, in which Lorentzian manifolds and causal sets stand on the same footing. Bombelli *et al.* introduce in [9] a closeness function, which is a distance when restricted to compact, distinguishing causal measure spaces. In Section 2 we introduce the framework and the function as in the paper, and then go on to explore just how to calculate this function and comment on its behaviour in various cases.

A good fraction of the time was spent exploring numerically properties of relatively large causets extracted from a background spacetime. Most of the results are already existent in the literature. A brief summary is presented in Appendix A.

1 Minkowski Spacetime from Causal Structure

We treat Minkowski spacetime as the pair $\mathcal{M} = (\mathbb{R}^4, \eta)$. The metric η defines a unique partial order on $\mathcal{M}$: the causal structure. We use the causal order to identify various geometrical object without ever invoking the metric or referring to the labels on the points. Ultimately, this leads to the definition of the vector space and of the metric up to a conformal factor, which completes the proof.

[1] In 4 dimensions, the metric has 10 free parameters, so we say the causal structure encodes $\frac{9}{10}$ of the metric.

Setup

Minkowski spacetime is the flat, empty spacetime of special relativity. No curvature; no gravity. In the context of general relativity it is a manifold isomorphic to $\mathbb{R}^4$, equipped with the Minkowski metric $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$. Because it is flat, however, we can model it as $\mathbb{R}^4$, with an affine vector space $\mathbb{V}$ and treat the metric η as a bilinear form:

$$\eta : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$$

- *Bilinear:* $\eta(\alpha u + \beta v, w) = \alpha\eta(u, w) + \beta\eta(v, w)$
- *Symmetric:* $\eta(u, v) = \eta(v, u)$
- *Non-degenerate:* $\eta(u, v) = 0, \forall u \implies v$ is the null vector. $\qquad \forall u, v, w \in \mathbb{V}, \forall \alpha, \beta \in \mathbb{R}$

Minkowski frames of reference have basis vectors $\{\hat{\mathbf{e}}_0, \hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ satisfying:

$$\eta(\hat{\mathbf{e}}_0, \hat{\mathbf{e}}_0) = -1 \qquad \eta(\hat{\mathbf{e}}_i, \hat{\mathbf{e}}_j) = \delta_{ij}$$

and

$$\eta(\hat{\mathbf{e}}_0, \hat{\mathbf{e}}_i) = 0$$

for $i, j = 1, 2, 3$. This ensures the equivalent component expression in any Minkowski frame:

$$\eta(u, v) := \eta_{\mu\nu} u^\mu v^\nu = -u^0 v^0 + \sum_{i=1}^{3} u^i v^i$$

1.1 Causal Structure

The causal structure is represented by a partial order:

Definition 1.1. A partial order on a set of points A is a binary relation '$\preceq$' with the following properties:

- *Reflexive*: $p \preceq p$
- *Acyclic*: $(p \preceq q) \wedge (q \preceq p) \implies p = q$
- *Transitive*: $(p \preceq q) \wedge (q \preceq r) \implies p \preceq r \qquad \forall p, q, r \in A$

Two event p causally precedes event q if and only if there exists a null or timelike future directed curve from p to q. Given two points in Minkowski spacetime, the existence of such a causal curve can be established at once using the metric: by looking at the *invariant interval*.

Definition 1.2. Given two points $p, q \in \mathcal{M}$, their invariant interval is

$$s^2(p, q) = \eta(u, u)$$

where $u \in \mathbb{V}$ is the displacement vector from p to q

A direct corollary is that, in a Lorentzian frame of reference:

$$s^2(p, q) = -(q^0 - p^0)^2 + \sum_{i=1}^{3} (q^i - p^i)^2$$

If the invariant interval between two points is positive, there exists a timelike curve joining them, if it is nought, then the curve is a null line. In these cases the points are causally connected. If the interval is positive, they are spacelike separated, causally disconnected.

Definition 1.3. We establish the partial order '$\preceq$' on $\mathcal{M}$ such that it satisfies, in any Lorentzian frame:

$$p \preceq q \iff (s^2(p,q) \leq 0) \wedge (p^0 \leq q^0) \tag{1.1}$$

One can start straight away to define objects using the causal structure, and see how this relates to shapes. To start, we define the *causal future* $J^+(p)$ and *causal past* $J^-(p)$ of a point p to be respectively the set of points in $\mathbb{R}^4$ that p precedes and that precede p. That is:

$$\begin{aligned} J^+(p) &= \{q \in \mathcal{M} \mid p \preceq q\} \\ J^-(p) &= \{q \in \mathcal{M} \mid q \preceq p\} \end{aligned} \tag{1.2}$$

These have an obvious *geometrical* meaning, $J^\pm(p)$ can be equivalently defined in terms of the metric:

$$\begin{aligned} J^+(p) &= \{q \in \mathcal{M} \mid (s^2(p,q) \leq 0) \wedge (q^0 \geq p^0)\} \\ J^-(p) &= \{q \in \mathcal{M} \mid (s^2(p,q) \leq 0) \wedge (q^0 \leq p^0)\} \end{aligned} \tag{1.3}$$

Taking the particular example where p is at the origin, all points $q \in J^+(p)$ satisfy the inequalityes:

$$0 \leq q^0 \leq \sum_{i=1}^{3} (q^i)^2 \tag{1.4}$$

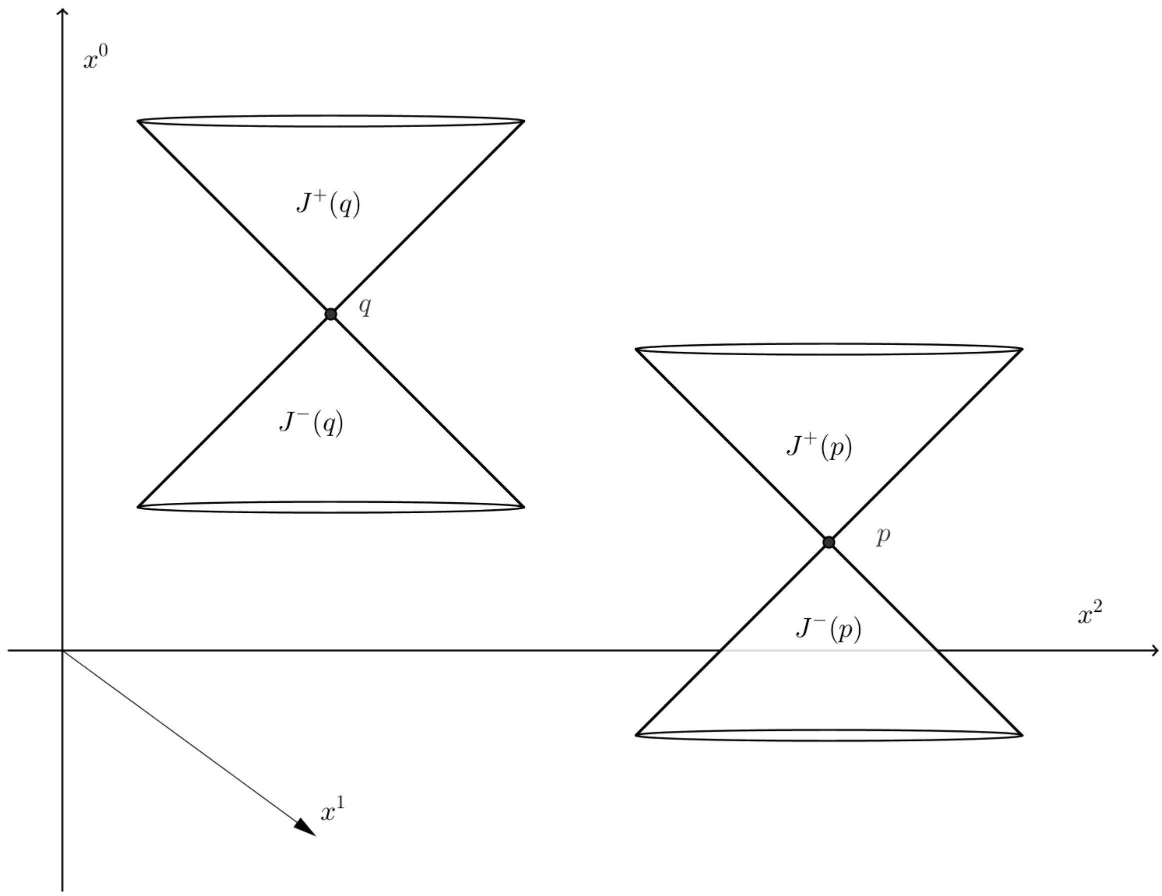

Figure 1: A 3 dimensional view of the light cones of two spacelike separated points.

We can now see the shape in spacetime of this subset: as the union of concentric balls of growing radius at various time co-ordinates, or as a cone with the tip on the origin in a 2+1 dimensional view of the spacetime, as in Figure 1. The four dimensional geometrical shape of $J^\pm$ is characteristic of Minkowski spacetime, and the set *looks* the same in all Minkowski frames of reference, i.e. to all inertial observers.

This is the starting point of the proof. We defined the partial order thanks to the metric. We can now formulate set theoretic definitions of objects using solely the pair $(\mathbb{R}^4, \preceq)$ as defined above, with no mention the labels on the points, nor the metric: just the ordering. We can then check what geometrical objects these causal objects correspond to using the full machinery of the vector space and metric.

We will first derive causal definitions for lines and planes, then use these to define the vector space and finally build a basis and defining the metric.

1.2 Null lines: the first step towards geometry

In Minkowski spacetime there exist three kinds of straight lines, spacelike, null and timelike, depending on the nature of their tangent vector. What are their causal properties?

Any two points in a spacelike line, as the name implies, are incommensurable under the causal relation[2]. Any two points on a null line are related under the causal relation, and this makes any null line a *totally ordered* set. The same is true for timelike lines. The causal relation might seem as being unable to distinguish between null and timelike lines.

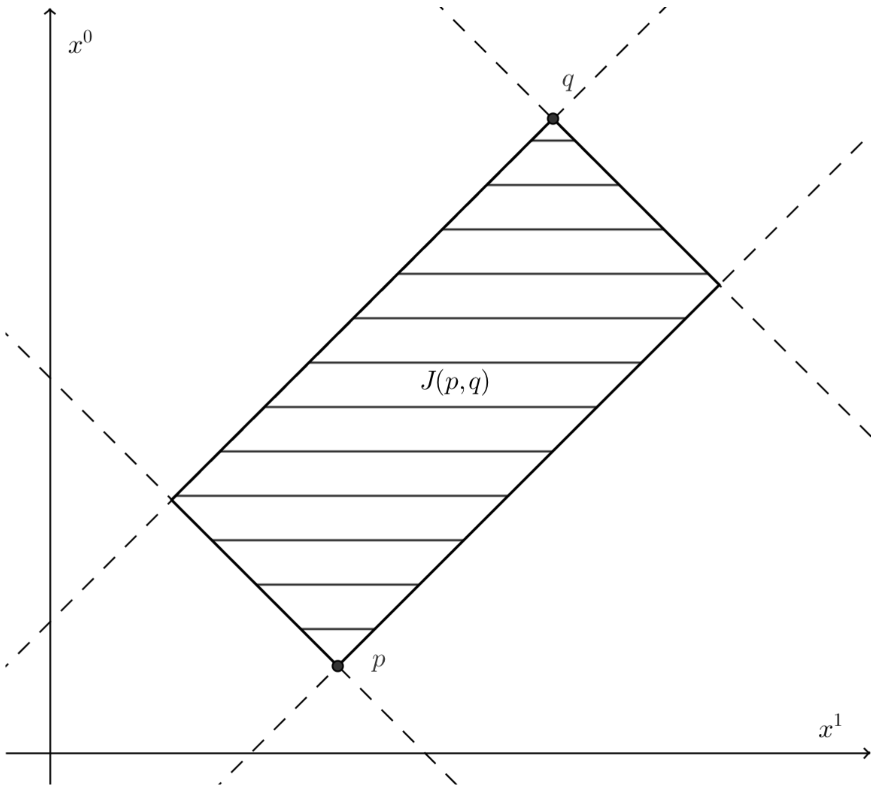

Figure 2: A view of the causal interval in a 2-dimensional slice.

Consider again causal past and future. We can create a new family of sets by intersecting pasts and futures of different points.

Definition 1.4. The causal interval between two points p and q is the set

$$J(p, q) = \{x \in \mathcal{M} \mid p \preceq r \preceq q\} \tag{1.5}$$

Again, this is a purely causal definition. The set of all causal intervals is defined purely in causal terms, using only $(\mathbb{R}^4, \preceq)$, with no reference to the labels on the points. Yet, we can see what shape they take geometrically, as in Figure 2.

[2] A spacelike line s will lie along one of the spacelike axes of some frame of reference. Say for example $s = \{p \in \mathcal{M} \mid p^\mu = (0, \lambda, 0, 0)\}$. It is fairly easy to see that any two points in s will not be related by the partial order. A similar thing can be done for timelike, and null lines. Selecting a particular frame often greatly simplifies calculations, with no loss of generality.

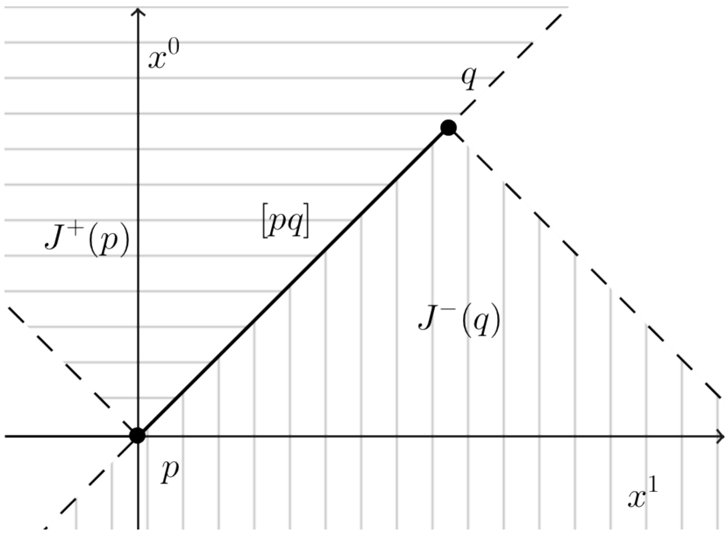

Figure 3: The segment $[pq]$ as a subset of the line (pq), and as the intersection of the past and future of the endpoints.

Let's look at the causal interval between two null separated points. Let p lie at the origin and $q^\mu = (q^0, q^0, 0, 0)$, as in Figure 3. From the 2+1 Minkowski sketch it looks like the causal interval $J(p,q)$ is indeed the null segment $[pq]$. Let's verify this:

Proof. The null segment is, in this frame, the set of points:

$$[pq] = \{x \in \mathcal{M} \mid x^\mu(\lambda) = (\lambda, \lambda, 0, 0), \forall \lambda \in [0, q^0]\}$$

For all such points $s^2(p,x) = s^2(x,q) = 0$ and $p \preceq x \preceq q \therefore x \in J(p,q)$. Thus the null segment is included in $J(p,q)$. What about points which are not on the segment? Consider the point $y^\mu = (\lambda, \lambda + \epsilon, 0, 0)$, for some $\lambda \in [0, q^0]$ and $0 < \epsilon$.

$$s^2(p,y) = 2\epsilon\lambda + \epsilon^2 > 0 \implies p \npreceq y$$

In the general case, $y^\mu = (\lambda, \lambda + \epsilon_1, \epsilon_2, \epsilon_3)$, $\epsilon_1, \epsilon_2, \epsilon_3 \in \mathbb{R}$. Then

$$s^2(p,y) = 2\epsilon_1\lambda + (\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2)$$
$$s^2(q,y) = 2\epsilon_1(\lambda - q) + (\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2)$$

So that either $p \npreceq y$ or $y \npreceq q$ unless $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0$. So null segment is indeed $J(p,q)$. Although the calculation was done in a particular frame, it is valid for any two null separated points because of the symmetries of Minkowski spacetime. Any null segment is therefore the causal interval of its endpoints. □

Null segments are totally ordered, so we learn that null segments are totally ordered causal intervals. This is a purely causal statement, as it only invokes the causal relation. We use it to define a causal null segment.

Definition 1.5. A *causal null segment* is a causal interval $J(p,q)$ such that, for any $x, y \in J(p,q)$,

$$J(x,y) \subseteq J(p,q)$$

We need to prove that all causal null segments are geometrically null segments.

Proof. Let $J(p,q)$ be a causal null segment, so that any two points $x, y \in J(p,q)$ are related. Say $x \preceq y$. If x and y are not null separated, then $J(x,y)$ contains spacelike points and thus is not totally ordered. However $J(x,y) \subset J(p,q)$, which leads to a contradiction. Thus any pair of points in a causal null segment is null separated, which implies they must be geometrically aligned. □

We have proved all causal null segments are geometrically null segments and conversely; null segments can be defined purely in causal terms. This leads to the definition of the null line in causal terms.

Definition 1.6. A causal null line l is an inextensible totally ordered set such that $J(p,q) \subset l, \forall p, q \in l$.

The causal relation seems barren: for any two pair of points, it will give one of three possible answers. There is no apparent sense of distance, or direction, which is natural when thinking about geometry. However, looking at overall causal properties allows to select subsets with specific geometric properties without invoking the coordinates at any point. The next step is another case where we do this explicitly.

1.3 Null Hyperplane

Having a causal definition of the null line, it is natural to consider its causal past and future. It turns out these are not, as one might expect, the whole of the spacetime. Given a null line l, what are the points spacelike to it?

Consider the null line through the origin and with tangent vector $n^\mu = (1,1,0,0)$:

$$l = \{x \in \mathcal{M} \mid x^\mu = (x^0, x^0, 0, 0)\}$$

We denote by $l^\natural$ the set of points in $\mathcal{M}$ not causally related to l:

$$\begin{aligned}
y \in l^\natural &\iff s^2(x,y) > 0, \forall x \in l \\
&\iff -(y^0 - x^0)^2 + (y^1 - x^0)^2 + (y^2)^2 + (y^3)^2 > 0, \forall x^0 \in \mathbb{R} \\
&\iff -(y^0)^2 + (y^1)^2 + (y^2)^2 + (y^3)^2 + 2x^0(y^0 - y^1) > 0, \forall x^0 \in \mathbb{R} \\
&\iff \big(y^0 = y^1\big) \wedge \big((y^2)^2 + (y^3)^2 > 0\big) \\
&\iff y^\mu = (y^0, y^0, y^2, y^3) \wedge \big(y^2 \neq 0 \vee y^3 \neq 0\big)
\end{aligned} \tag{1.6}$$

In particular, we see that the set of points $l \cup l^\natural$ is what is geometrically called a null hyperplane: a subset in which all curves are normal to a particular null vector. The null hyperplane H through a point p with normal vector n^μ is defined as:

$$H = \{x \in \mathcal{M} \mid \eta(n^\mu, (x-p)^\mu) = 0\} \tag{1.7}$$

There is always a frame where p is at the origin and $n^\mu = (1,1,0,0)$. The coordinates of the points in the hyperplane are then:

$$H : x^\mu = (x^0, x^0, x^2, x^3) \tag{1.8}$$

We observe that H contains many null lines tangent to n^μ. Since we can always choose a frame in which the co-ordinates of the points of the frame take this form, we recognise at once that we can define a null hyperplane causally:

Definition 1.7. Given a null line l, a null hyperplane is the set $l \cup l^\natural$.

1.4 Spacelike Lines

Looking at equation (1.8), we can see that any straight line in a null hyperplane is necessarily null or spacelike. Let's see what happens when intersecting three null hyperplanes.

We choose for simplicity of calculation the three null hyperplanes H_1, H_2, H_3, containing the origin and having as normal vectors:

$$n^\mu = (1,1,0,0) \qquad m^\mu = (1,-1,0,0) \qquad k^\mu = (1,0,1,0)$$

respectively. The points in these planes have the following coordinates:

$$H_1 : x^\mu = (x^0, x^0, x^2, x^3) \qquad H_2 : x^\mu = (x^0, -x^0, x^2, x^3) \qquad H_3 : x^\mu = (x^0, x^1, x^0, x^3) \tag{1.9}$$

We see that intersecting first H_1 and H_2, we obtain a subset where all points are spacelike.

$$H_1 \cap H_2 : x^\mu = (0, 0, x^2, x^3)$$

This is in fact a spacelike 2-plane, which we will discuss in more detail in the next section. Intersecting H_3 as well leaves us with the line $s = \{x \in \mathcal{M} \mid x^\mu = (0, 0, 0, x^3)\}$: a spacelike line. This leads to the definition of a causal spacelike line.

Definition 1.8. A causal spacelike line is the intersection of three non-disjoint, non-identical causal null hyperplanes.

All causal spacelike lines are geometrically null lines, and vice versa.

Proof. Consider three null hyperplanes H_1, H_2, H_3 intersecting at the origin. Without loss of generality we can choose them to have

$$n^\mu = (1, 1, 0, 0) \qquad m^\mu = (1, \hat{\mathbf{m}}) \qquad k^\mu = (1, \hat{\mathbf{k}}) \tag{1.10}$$

respectively as normal vectors, where $\hat{\mathbf{m}} \neq \hat{\mathbf{k}}$ are unit vectors. Intersecting the three hyperplanes results in three orthogonality conditions. The one imposed by H_1 on the coordinates of a point x is $x^0 = x^1$. The other two conditions are $\hat{\mathbf{m}} \cdot \mathbf{x} = x^1$ and $\hat{\mathbf{k}} \cdot \mathbf{x} = x^1$, or equivalently,

$$\begin{cases} x^1(m^1 - 1) + x^2 m^2 + x^3 m^3 = 0 \\ x^1(k^1 - 1) + x^2 k^2 + x^3 k^3 = 0 \end{cases}$$

These are two linear equations for three variables. The set of solutions is a line in $\mathbb{R}^3 : \mathbf{x} = \lambda \mathrm{s}$ where $\lambda \in \mathbb{R}$. Thus $x \in P_1 \cap P_2 \cap P_3 \iff x^\mu = (\lambda s^1, \lambda s^1, \lambda s^2, \lambda s^3)$, a spacelike line. A causal spacelike line is geometrically a spacelike line.

A given geometrical null line always passes through the origin of a particular frame of reference, and is along the third spacelike axis. We already showed how this line is the intersection of three null hyperplanes. A geometrical spacelike line is a causal spacelike line. □

1.5 2-Planes

2-Planes in Minkowski spacetime are defined geometrically with a point and two mutually orthogonal normal vectors. There turns out to be three kinds of 2-planes (see Figure 4):

- *Spacelike plane:* one spacelike and one timelike normal vectors, points are spacelike.
- *Null plane:* one null and one spacelike normal vectors, points are spacelike or null.
- *Timelike plane:* two spacelike normal vectors, points are either spacelike, null or timelike.

We will express the equation of one exemplar of each kind of plane, to illustrate its causal content and motivate the causal definition, which consists of intersection of known causal objects.

Spacelike 2-Plane

We have already shown that by intersecting two causal null hyperplanes we obtained a spacelike 2-plane.

Definition 1.9. A causal spacelike 2-plane is the intersection of two non-disjoint, non-identical, null hyperplanes.

We prove that causal spacelike 2-planes are geometrical null 2-planes by showing that being normal to two null vectors is equivalent to being normal to a pair of orthogonal time and space vectors.

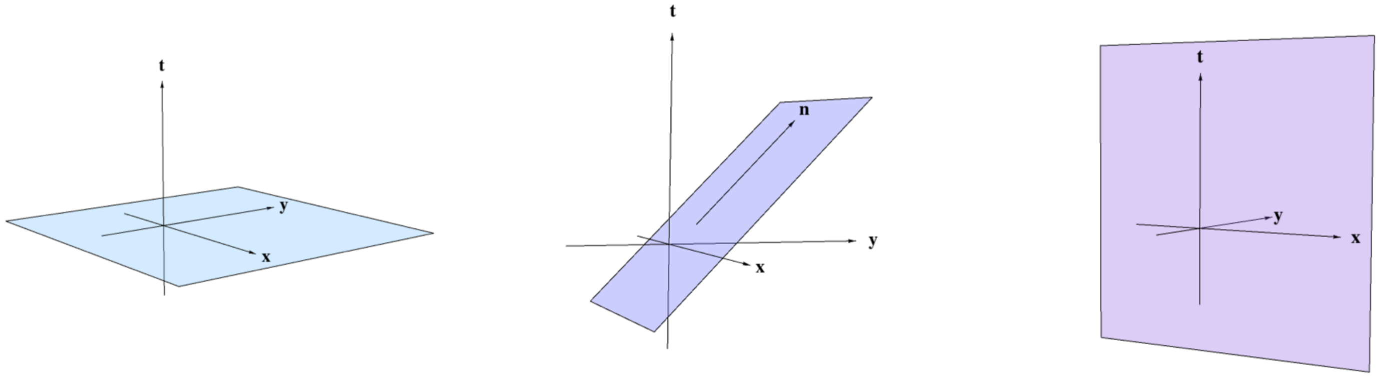

Figure 4: The three kind of 2-planes, respectively, Spacelike, Null and Timelike

Proof. Given pair of null vectors $n^\mu = (a, \mathbf{a})$ and $m^\mu = (b, \mathbf{b})$ where $a = |\mathbf{a}|$ and $b = |\mathbf{b}|$, we consider the following linear combinations

$$u^\mu = n^\mu + m^\mu \qquad v^\mu = n^\mu - m^\mu$$

It follows

$$\begin{cases} u^\mu u_\mu = 2n^\mu m_\mu = 2(\mathbf{a}\cdot\mathbf{b} - a^0 b^0) < 0 \\ v^\mu v_\mu = -2n^\mu m_\mu > 0 \end{cases}$$

which means that u^μ is timelike and v^μ is spacelike. Additionally, $u^\mu v_\mu = 0$, so that these are indeed two orthogonal time and space vectors. □

Null 2-Plane

Geometrically, a null 2-plane is defined using one null and one spacelike normal vectors. We know null hyperplanes have a normal null vector, thus null 2-planes too are subsets of null-hyperplanes.

The two normal vectors are taken to be orthogonal and, in some frame, they are $n^\mu = (1, 1, 0, 0)$ and $m^\mu = (0, 0, 0, 1)$. The resulting null 2-plane has the coordinates: $x^\mu = (x^0, x^0, x^2, 0)$. So the null 2-plane contains a series of null lines tangent to n^μ, translated in the direction of m^μ. Any two null lines in the null 2-plane are joined by spacelike lines lying in the 2-plane.

Definition 1.10. A causal null 2-plane is the union of two null lines in a null hyperplane, and the set of all spacelike lines intersecting both of them.

Timelike 2-Plane

A timelike 2-plane is geometrically defined with two spacelike normal vectors. These are mutually orthonormal so they are always the two basis vectors $\hat{\mathbf{e}}_2$ and $\hat{\mathbf{e}}_3$ of some frame. The resulting subspace is one that is most familiar to Physics undergraduates, the 1+1 dimensional Minkowski spacetime, namely, the subspace with coordinates:

$$x^\mu = (x^0, x^1, 0, 0)$$

In a fashion similar to the null 2-plane, one can construct this particular timelike 2-plane as the union of all the spacelike lines joining the two intersecting null lines $\{x^\mu = (x^0, x^0, 0, 0)\}$ and $\{y^\mu = (y^0, -y^0, 0, 0)\}$, and the point where they intersect. This bring us to the definition.

Definition 1.11. A causal timelike 2-plane is the union of two intersecting causal null lines, and all spacelike lines joining them.

Causal timelike 2-planes are geometrical 2-planes because any two intersecting null lines always lie in a geometrical timelike 2-plane.

Proof. Consider two null lines l and d intersecting at point p and with tangent vectors n^μ and m^μ respectively. There exists a frame in which p lies at the origin and the tangent vectors can be taken to be: $n^\mu = (1, 1, 0, 0)$ and $m^\mu = (1, m^1, m^2, 0)$. In that frame, the two vectors

$$a^\mu = (0, 0, 0, a^3) \qquad b^\mu = \left(\frac{m^2}{1 - m^1}, \frac{m^2}{1 - m^1}, 1, 0 \right)$$

are two orthonormal space vectors that are normal to both tangent vectors n^μ and m^μ. They thus define a timelike plane through p which includes l and d. In some frame, the vectors a^μ and b^μ are the basis vectors. □

Final Touches

So far, we have found causal definitions for null and spacelike lines, and all the 2-planes, we are only missing the timelike line.

Definition 1.12. Causal timelike lines are intersections of causal timelike 2-planes

We also define segments. We already have null segments. Timelike segments are defined as subsets of timelike lines.

Definition 1.13. Given a timelike line t, the segment $[pq] \subset t$, where $p \preceq q$ is:

$$[pq] = t \cap J(p, q)$$

The causal relation offers a notion of betweenness in totally ordered sets. Points on a spacelike line are not causally related though, so the definition is different (see Figure 5):

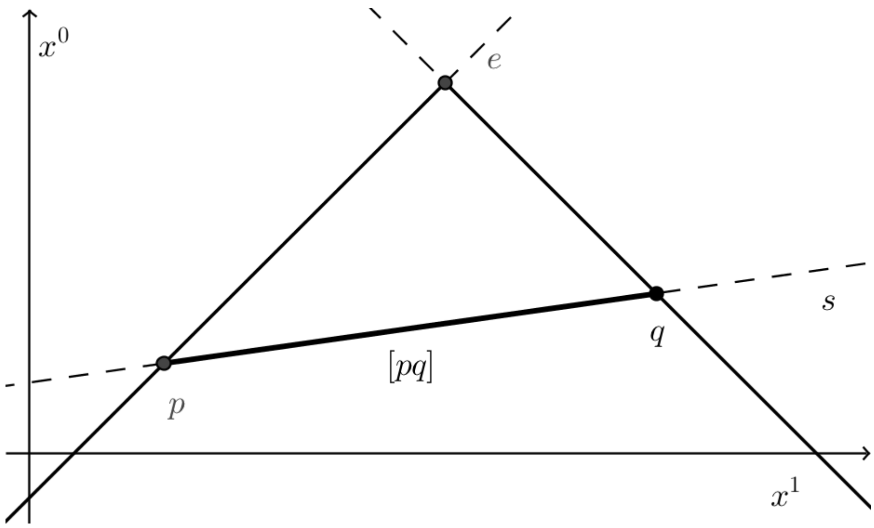

Figure 5: The intersection of s with the past of e is exactly the segment $[pq]$.

Definition 1.14. Given a spacelike line s and two points $p, q \in s$, the spacelike segment $[pq]$ is the set

$$[pq] = s \cap J^-(e)$$

where e is a point which is null and to the future to both p and q.

1.6 Plane Geometry

We can now talk about planes, lines and segments in a purely causal fashion, never having to invoke the metric. But some things are still missing. Although we are able to tell if two given lines intersect, we have little notion of their relative direction.

Planes offer an easy definition of parallel lines:

Definition 1.15. Parallel lines are lines lying in the same plane that do not intersect. Parallel segments are segments whose lines are parallel.

With which we can then make parallelograms.

Definition 1.16. $abcd$ is a causal parallelogram if $[ab]$ is parallel to $[dc]$ and $[bc]$ is parallel to $[ad]$.

1.7 The Vector Space

Having causal definitions of parallelograms, we can use them to construct a vector space. We build the affine vector space out of equivalence classes of pairs of points.

Definition 1.17. The equivalence relation '$\backsim$' is established in the following manner:

$$(x, y) \backsim (a, b) \iff \begin{cases} (x = a) \wedge (y = b), & \text{or} \\ xyba \text{ is a parallelogram} \end{cases} \tag{1.11}$$

This is indeed a good equivalence relation:

Proof.

1. *Reflexivity:* $(x, y) \backsim (x, y)$

   By definition.

2. *Symmetry:* $(x, y) \backsim (a, b) \iff (a, b) \backsim (x, y)$

   Since if $xyba$ is a parallelogram then $abyx$ is a parallelogram too.

3. *Transitivity* $(x, y) \backsim (a, b), (a, b) \backsim (c, d) \Rightarrow (x, y) \backsim (c, d)$

   If $xyba$ and $abdc$ are parallelograms, then $xydc$ is a parallelogram too.

□

This partitions the set of ordered pairs of points $\mathcal{M} \times \mathcal{M}$ in equivalence classes:

Definition 1.18.

$$[(x, y)] := \{(x', y') \in \mathcal{M} \times \mathcal{M} : (x', y') \backsim (x, y)\} \tag{1.12}$$

The set of which will be the causal vector space $\mathbb{V}_C$:

Definition 1.19.

$$\mathbb{V}_C := \{[(x, y)] : (x, y) \in \mathcal{M} \times \mathcal{M}\} = \frac{\mathcal{M} \times \mathcal{M}}{\backsim} \tag{1.13}$$

Moving on, we need to equip $\mathbb{V}_C$ with vector addition and scalar multiplication.

Definition 1.20. Consider two vectors $u = [(a, b)]$ and $v = [(b, c)]$. We define vector addition between elements in $\mathbb{V}_C$:

$$u + v = [(a, c)]$$

This operation defines a unique sum:

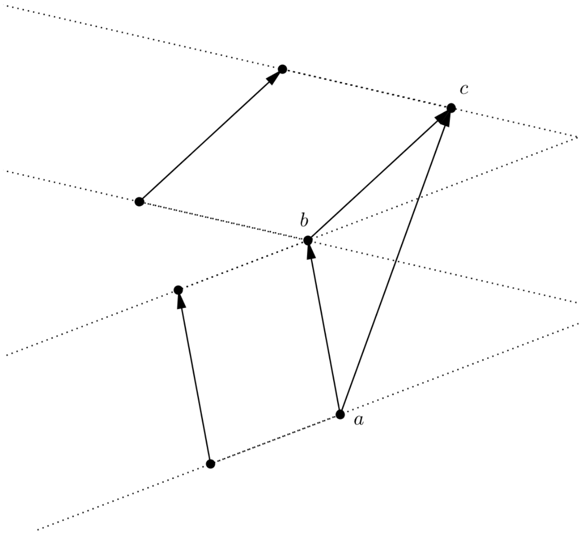

Figure 6: Equivalent pairs of points and affine vector addition.

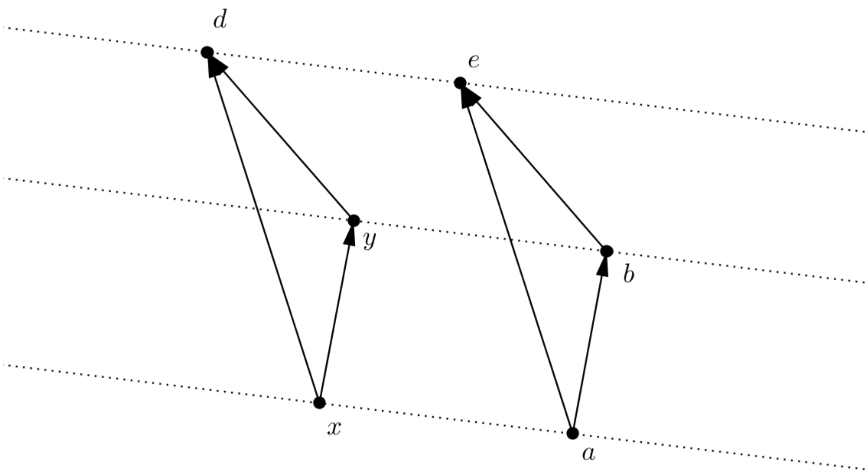

Figure 7: Uniqueness of the vector addition.

Proof. Assume $u = [(x, y)] = [(a, b)]$ and $v = [(y, d)] = [(b, e)]$. Showing $u + v$ is uniquely defined is showing

$$[(x, d)] = [(a, e)]$$

We note that $xyba$ is a parallelogram, so that

$$(x, a) \backsim (y, b)$$

Also $ydeb$ is a parallelogram so that

$$(y, b) \backsim (d, e)$$

By transitivity then

$$(x, a) \backsim (d, e)$$

which implies that $edxa$ is a parallelogram and therefore

$$(x, d) \backsim (a, e)$$

$\square$

We also need to show this addition has the correct properties of vector addition.

Proof. 1. *Commutativity:* $u + v = v + u$

Say $[(a,b)] + [(b,c)] = [(a,c)]$ and $[(b,c)] + [(a,b)] = [(b,d)]$, where $[(c,d)] = [(a,b)]$ for some point d. Since $[(c,d)] = [(a,b)]$, $abdc$ is a parallelogram, therefore $[(a,c)] = [(b,d)]$.

2. *Associativity:* $(u + v) + w = u + (v + w)$

$$([(a,b)] + [(b,c)]) + [(c,d)] = [(a,b)] + ([(b,c)] + [(c,d)]) = [(a,d)]$$

3. *Zero Elelement:* $o + v = v$

The zero element is the equivalence class $[(a,a)]$, where $a \in \mathcal{M}$.

4. *Inverse* $u + (-u) = o$

$-[(a,b)] = [(b,a)]$ $\square$

Addition can be used to define vector multiplication by a scalar. This is a sketch of how it is done.

- Integers $\mathbb{Z}$

  If $n \in \mathbb{N}$, then
  $$nu = u + u + \cdots$$
  with n terms.

  If $m = -n$, then
  $$mu = n(-u)$$
  This is operation easily seen to be *commutative* and *associative*. Multiplication by 0 returns the vector $[(a,a)]$ and the multiplicative identity is 1.

- Rationals $\mathbb{Q}$

  Let's start by defining multiplication by fractions of 1.

  For $q = \frac{1}{n}$ where $n \in \mathbb{N}\backslash\{0\}$,
  $$v = \frac{1}{n}u \iff nv = u$$
  This operation is *commutative*:
  $$v = \frac{1}{m}\left(\frac{1}{n}u\right) \iff mv = \frac{1}{n}u \iff (nm)v = u \iff v = \frac{1}{n}\left(\frac{1}{m}u\right)$$
  $$\iff v = \frac{1}{nm}u$$
  and *associative*:
  $$w = \frac{1}{n}(u + v) \iff nw = u + v \iff w = \frac{1}{n}u + \frac{1}{n}v$$
  Multiplication by the set of rationals is achieved by combining multiplication by integer and multiplication by fractions of unity.

- Reals $\mathbb{R}$

  Multiplication by irrationals is done in a method analogue to Dedekind cuts.

This completes the construction of the vector field. Note now that we are getting closer to a causal sense of relative lengths, but we can only compare lengths of colinear vectors, and we have yet no sense of angles. Achieving this is the final step of the proof.

1.8 Causally Constructing a Minkowski Basis

We first causally define a set of very useful points: the causal space ball.

Definition 1.21. The causal space ball centred at point a, orthogonal to the future directed timelike vector $u = [(a, c)] \in \mathbb{V}_C$, is

$$B(a,u) = \left\{b : \exists b' \mid [(a,b)] = -[(a,b')] \text{ and } b, b' \in LC^-(c)\right\} \tag{1.14}$$

where $LC^-(c)$ denotes the past lightcone[3] of c.

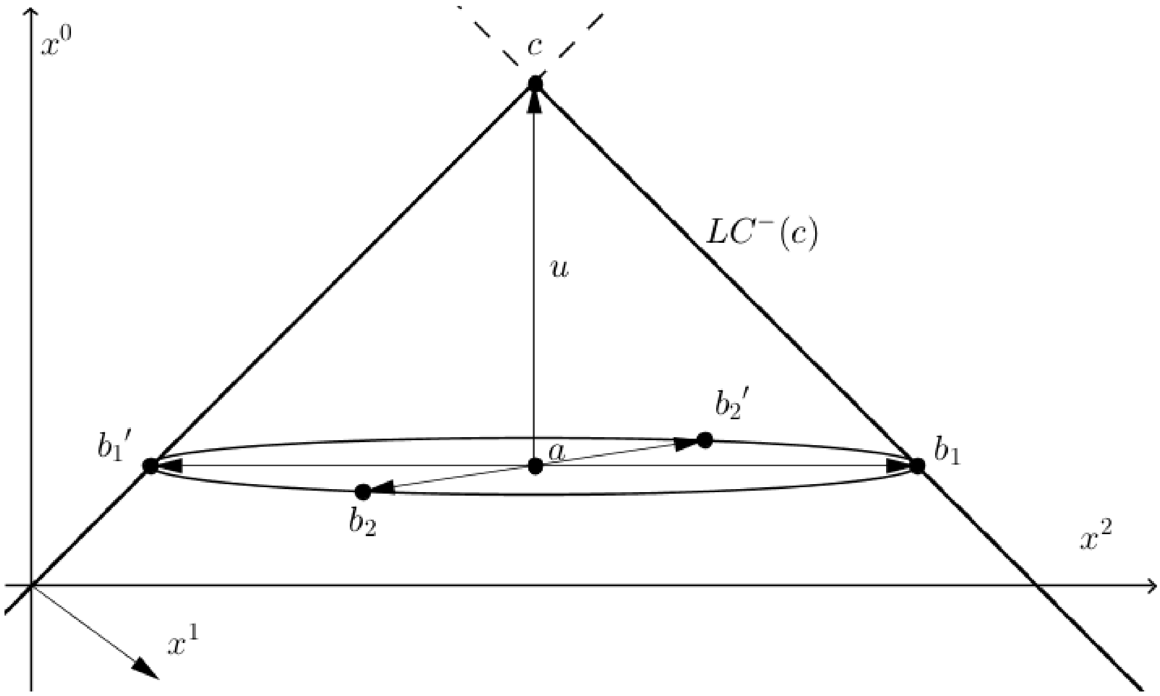

Figure 8: A 2+1 view of the construction of the space ball, where it looks like a circle.

This is again a purely causal definition, and this object has interesting geometrical properties. For a causal space ball $B(a, u)$, all vectors $[(a, b)]$, such that $b \in B(a, u)$, are geometrically orthogonal to the vector u, and have opposite norm to it.

Proof. Let $u^\mu = [(a, c)]$ and $v^\mu = [(a, b)]$ so that $-v^\mu = [(a, b')]$ for $b, b' \in B\,(a, u)$.

From the requirement that b and b' are null to c, we have:

$$\begin{cases} (v^\mu - u^\mu)^2 = \eta_{\mu\nu}(v^\mu v^\nu + u^\mu u^\nu - 2v^\mu u^\nu) = 0 \\ (-v^\mu - u^\mu)^2 = \eta_{\mu\nu}(v^\mu v^\nu + u^\mu u^\nu + 2v^\mu u^\nu) = 0 \end{cases}$$

which together imply:

$$\eta_{\mu\nu} v^\mu u^\nu = 0 \qquad \eta_{\mu\nu} v^\mu v^\nu = -\eta_{\mu\nu} u^\mu u^\nu$$

□

In other words, a space ball is geometrically a sphere of points in a spacelike hyperplane. Incidentally, spheres give us an opportunity to find perpendicular lines, which we will get at once. Spacelike hyperplanes in Minkowski spacetime are Euclidean, and in Euclidean geometry lines tangent to a sphere are perpendicular to the radius of the sphere at the tangent point. There is a straightforward causal definition.

Definition 1.22. A line l tangent to a causal space ball B is such that $l \cap B$ is one single point.

[3] The set of points preceding and null to c, a causal object.

Construction

We wish to causally construct a Minkowski basis centred at a point p. We start from a future directed causal timelike vector $\hat{\mathbf{e}}_0$.

We then need three spacelike vectors with identical norm. Start by constructing the ball $B_1 = B(p, \hat{\mathbf{e}}_0)$. Selecting a point $b \in B_1$ results in a vector $[(p, b)]$ which is geometrically orthonormal to $\hat{\mathbf{e}}_0$. We take this to be our first spacelike basis vector,

$$\hat{\mathbf{e}}_1 = [(p, b)]$$

Let now l_1 be the spacelike line through a and b. Let also l_2 be the tangent to B_1 at point b. The spacelike lines l_1 and l_2 are thus (causally and geometrically) perpendicular. We define our second spacelike basis vector:

$$\hat{\mathbf{e}}_2 = [(b, c)] = [(p, b_1)]$$

where $c \in l_2$, ensuring $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ are geometrically orthogonal, and $b_1 \in B_1$, ensuring they have the same norm.

The third orthonormal vector is found by constructing a second space ball $B_2 = B(q, \hat{\mathbf{e}}_0)$, where the point q is such that l_1 and l_2 are both tangent to B_2. The vector $[(c, q)]$ is then our third and last spacelike basis vector, $\hat{\mathbf{e}}_3$.

The result of this construction is a basis $\{\hat{\mathbf{e}}_0, \hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$, determined purely in causal terms, where the vectors have the geometrical property of being mutually orthogonal, and have same norm.

1.9 The Metric

Once we have chosen a particular basis $\{\hat{\mathbf{e}}_0, \hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ for the causal vector space, we can finally causally define the metric.

Definition 1.23. The causal metric is the bilinear form

$$\eta_C : \mathbb{V}_C \times \mathbb{V}_C \to \mathbb{R}$$

with the values:

$$\eta_C(\hat{\mathbf{e}}_0, \hat{\mathbf{e}}_0) = -1 \qquad \eta_C(\hat{\mathbf{e}}_i, \hat{\mathbf{e}}_j) = \delta_{ij}$$

and

$$\eta_C(\hat{\mathbf{e}}_0, \hat{\mathbf{e}}_i) = 0$$

for $i, j = 1, 2, 3$.

The construction of the basis and metric can be done purely in causal terms. Geometrically, we know that the basis spans the whole set $\mathcal{M}$, so that now we can construct bases at any point using the causal metric η_C. We can use $\mathbb{V}_C$ to define curves, derivatives and, in fact, the whole of Special Relativity.

Note that the construction of the basis is not unique: there is freedom in choosing the timelike vector, which reflects the geometrical boost symmetries of Minkowski spacetime, freedom in the first two spacelike basis vector, corresponding to the three spatial rotations, and freedom in choosing the last spacelike vector, reflecting parity symmetry. Additionally, it is impossible to causally determine the length of the vector $\hat{\mathbf{e}}_0$. We can set it to be of length 1, and this corresponds to fixing to the conformal factor.

Conclusion

Given Minkowski spacetime, we have proved that having the causal structure is sufficient to reconstruct the affine vector space, equipped with a metric that is conformally equal to the original. This is a proof of the Malament result, in the special case of a flat spacetime. The flatness is fundamental in the construction of this particular proof. The more general proof involves more sophisticated mathematics.

2 A distance function for causets and manifolds

In the context of Causal Set Theory, adequately coarse grained causets are expected to approach spacetime manifolds [3]; having a measure of closeness between causets and manifolds is a necessity. A measure exists: the *probability* that a given causet arises from sprinkling on a manifold. Its probabilistic nature however rends it unpractical. As Bombelli *et al.* point out in a pre-print [9], both causal sets and Lorentzian manifolds are part of a larger family of sets called Causal Measure Spaces, which allows to put them on the same footing.

Most importantly they introduce a Gromov-Haussdorf function d_{GH}: a pseudo-distance on the space of causal measure spaces, and prove it is positive definite when restricted to compact spaces in which all points have distinct causal past and future.

In this section, we introduce causal measure spaces and d_{GH} as in [9]. We then take the function and start studying it quantitatively and qualitatively. We explicitly use it to calculate the distances between simple causets, prove that it has the expected behaviour when comparing a causet which approaches a one dimensional Lorentzian manifold, start to study its positive definiteness on the space of causets and, finally, comment on its relation with the graph distance. All the way the proofs will have a constructive feel, so that the behaviour of the function is exposed.

2.1 Causal Measure Spaces and a measure of closeness

Definition 2.1. A Causal Measure Space is a quadruple $(X, \prec, \mathcal{A}, \mu)$: a set X endowed with a strict partial order $\prec$, an associated σ-algebra $\mathcal{A}$ and a measure function μ on it, such that, for any point $x \in X$, the sets

$$I^+(x) = \{z \in X | x \prec z\} \qquad I^-(x) = \{z \in X | z \prec x\}$$

are elements of $\mathcal{A}$.

Note here we use a different symbol for the partial order. This is to denote a *strict* partial order, one which is *irreflexive* $(x \not\prec x)$. The two choices are equivalent, this is just a matter of notational convenience in the following work.

As stated, Lorentzian manifolds and causets are examples of causal measure spaces:

- In a chronological Lorentzian manifold (M, g), the partial order is that of the causal structure, the measure space is that of open sets measured using the volume element from the metric g. Two distinguishing Lorentzian Manifolds are isomorphic if and only if their causal measure spaces are.
- For a causet, $\mathcal{A} = 2^X$ and the measure can be a weighted counting measure: $\mu(B) \propto |B|, \forall B \subseteq X$. In this case, the requirement of distinguishability is lifted as it would eliminate many causets[4].

One can equip a causal measure space with a distance function compatible with the partial order.

Definition 2.2. For any causal measure space $(X, \prec, \mathcal{A}, \mu)$, we introduce the Lorentzian distance, a map $d : X \times X \to \mathbb{R}^+$:

$$d(x, y) = \begin{cases} \mu\left(A(x, y) \cup \{y\}\right) & \text{if } x \prec y \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

It has the following properties:

- *One-sidedness:* $d(x, y) > 0 \implies d(y, x) = 0 \qquad (NB : d(x, x) = 0)$
- *Reverse triangle inequality:* $d(x, y)d(y, z) > 0 \implies d(x, y) + d(y, z) \leq d(x, z)$

[4]In the discrete case, in contrast with the continuous case, the existence of non distinguishing elements does not imply closed, or almost closed, causal curves.

- *Compatibility:* If $d(x, y) > 0 \iff x \prec y$ (strong compatibility)

Any function with these three properties is a Lorentzian distance. There exist other choices of Lorentzian distance. This particular choice is motivated by the fact that the definition is valid on all causal measure spaces. Note in the case of causets, if the two elements are linked[5], their distance is the measure of a single element.

From the result of Malament, we know that two Lorentzian manifolds are conformally identical if their causal structure is the same. And since the Lorentzian distance uniquely determines the causal structure, two Lorentzian manifolds having isomorphic Lorentzian distances are conformally identical. Thus on wanting to measure the closeness of two causal measure spaces it is reasonable to compare their Lorentzian distances.

Definition 2.3. Consider two causal measure spaces $C_X = (X, \prec_X, \mathcal{A}_X, \mu_X)$ and $C_Y = (Y, \prec_Y, \mathcal{A}_Y, \mu_Y)$ equipped with Lorentzian distances and a pair of maps $f : X \to Y$ and $g : Y \to X$. We define the pairwise discrepancy for elements $x_1, x_2 \in X$:

$$\delta_f(x_1, x_2) := |d_X(x_1, x_2) - d_Y(f(x_1), f(x_2))| \tag{2.2}$$

And we can look at how well the map does overall by considering the worst case discrepancy:

$$\Delta_f := \sup\{\delta_f(x_1, x_2) \mid x_1, x_2 \in X\} \tag{2.3}$$

And analogous definitions for δ_g and Δ_g. These quantities give a measure of how good (f, g) is at preserving the Lorentzian distance. We say C_X and C_Y are 'ϵ-close' with respect to (f, g) if both Δ_f and Δ_g are less than or equal to[6] ϵ. This considering the shear number of possible maps from a set to another, this is really a statement about the maps rather than the spaces proper. However, if two spaces are very similar, we expect there exists a way of mapping them such that the discrepancy is really low, and, conversely. So the best map should offer a measure of closeness.

Definition 2.4. The Gromov-Hausdorff closeness function d_{GH} is a map from two causal measure spaces C_X and C_Y to the real numbers, defined as

$$d_{GH}(C_X, C_Y) := \inf\{\epsilon \mid C_X, C_Y \text{ are } \epsilon\text{-close}\}$$

This is the function that we will study in the rest of the work. It is easy to see it is symmetric and that it satisfies the triangle inequality. It is thus a pseudo-distance on the set of causal measure spaces. It measures how well the two spaces allow their Lorentzian distances to be mapped. Since there is no requirement on the map, d_{GH} indeed does enable to compare any two causal measure spaces and, in particular, a causet with a manifold, as promised.

In the paper, Bombelli *et al.* go on to prove that it is indeed a distance when restricted to some compact spaces. We will take another direction and explicitly explore its behaviour. Since d_{GH} is a statement about all possible maps, it is not in general straightforward to calculate it, while it is always possible to give upper bounds by calculating the discrepancy of a pair of maps. This is is particularly relevant when comparing two causets. Large causets are often very non-analytical complex networks. There is no definite method of finding the maps with lowest discrepancy. Or to show, once found a satisfactory map, that it is the best one. In some cases it is possible to find an intrinsic lowest value.

Let's start with the simple case of two chain causets.

2.2 Chains with same measure

Let[7]

$$\begin{cases} C_N = \{x_i \mid x_i \prec x_j \iff i < j, \forall i, j = 0, 1, ..., N-1\} \\ C_{N+1} = \{y_i \mid y_i \prec y_j \iff i < j, \forall i, j = 0, 1, ..., N\} \end{cases}$$

[5] Linked elements are such that $I(x, y) = \emptyset$. We also say x covers y.

[6] N.B. In [9], this is a strict inequality. In the scope of this paper, there is no substantial difference apart from simpler language.

[7] In the rest of the paper, we abuse the notation and refer to the causal measure space by the name of its set of points.

be endowed with a counting measure such that $\mu(\{x_i\}) = \mu(\{y_j\}) = 1$. The Lorentzian distances are then:

$$\begin{cases} d_{C_N}(x_i, x_j) = j - i \\ d_{C_{N+1}}(y_i, y_j) = j - i \end{cases} \tag{2.4}$$

if $i < j$. We are now to choose the maps. There are $(N+1)^N \cdot N^{N+1}$ possible pairs of maps. These are spaces with very similar causal structure, and most of the maps will not respect it. Since C_N is indeed a subset of C_{N+1}, a natural choice of maps seems:

$$\begin{aligned} f: &\quad x_i \to f(x_i) = y_i, \ \ \forall i = 0, 1, ..., N-1 \\ g: &\quad y_i \to g(y_i) = x_i, \ \ \forall i = 0, 1, ..., N-1 \text{ and } f(y_N) = x_{N-1} \end{aligned}$$

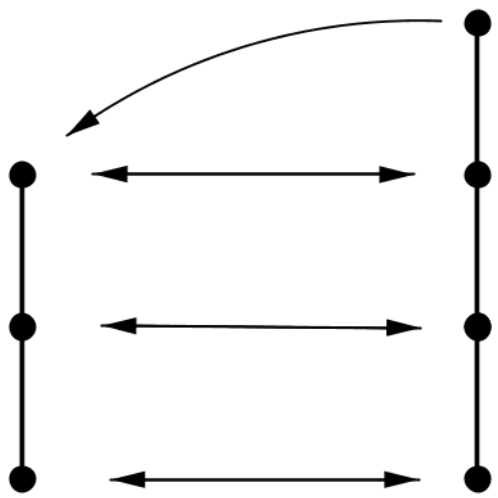

Figure 9: Hasse diagram of the 3-chain and 4-chain, along with one of the two pair of maps with which they are 1-close.

From the definitions then, $\delta_f(x_i, x_j) = 0$ identically, so that $\Delta_f = 0$. This is a good result: C_N is a subset of C_{N+1} and can be mapped with no discrepancy. The value of d_{GH} will depend on how well g does its mapping. In fact, $\delta_g(y_i, y_j) = 0$ for $i, j \leq N-1$. But suppose $i \leq N-1$ and $j = N$ then

$$\delta_g(y_i, y_N) = |N - i - (N - 1 - i)| = 1$$

Due to the one-sidedness of d_{C_N} and $d_{C_{N+1}}$, the remaining cases are nought. Therefore, $\Delta_g = 1$ and C_N and C_{N+1} are 1-close under this choice of maps.

Is there a better choice for g? The similar choice $g(y_i) = x_{i-1}$ and $g(y_0) = x_0$ gives the same result because

$$\delta_g(y_0, y_i) = |i - (i-1)| = 1$$

We can try mapping the element y_N to x_k for some k. We immediately see

$$\begin{cases} \delta_g(y_i, y_N) = |N - i - (k - i)| = N - k & \text{if } k > i \\ \delta_g(y_i, y_N) = N - i & \text{if } k < i \end{cases}$$

This is a worse result, because the map does not respect the similarity of the causets. We could imagine further jumbling up of the sets, mapping other elements of C_{N+1} to elements of C_N with very different causal futures and pasts, but it is evident this will not make a better map. We can try something more respectful of the causal order. For example, we can try:

$$\begin{cases} g(y_i) = x_i & \text{if } i \leq k \\ g(y_i) = x_{i-1} & \text{if } i > k \end{cases}$$

Here, $\delta_g(y_i, y_j) = 0$ for $i, j \leq k$ or $k < i, j$ but, again $\delta_g(y_i, y_j) = 1$ if $i \leq k < j$ so that $\Delta_g = 1$ again. We can actually prove that is the best and thus that $d_{GH}(C_N, C_{N+1}) = 1$.

Proof. Since one space is larger than the other, all maps will have at least two elements y_i and $y_{i'}$ such that $g(y_i) = g(y_{i'})$. Suppose $i < i'$, then

$$\delta_g(y_i, y_{i'}) = d_{C_{N+1}}(y_i, y_{i'}) \geq 1$$

Δ_g is at best as small as this, no matter the map. We have already found a map g such that $\Delta_g = 1$, and we have proved that there exists a map f such that $\Delta_f = 0$. Therefore

$$d_{GH}(C_N, C_{N+1}) = 1 \qquad \square$$

Consider now the case of a N-chain and an M-chain, for which there are $M^N \cdot N^M$ possible maps. Suppose $N < M$. A simple pair of maps is identical to the previous case, with the modification:

$$g(y_i) = x_{N-1} \qquad \forall i = N, N+1, ..., M$$

These maps show that the two chain are $(M - N)$-close. This choice of map for g seems a bit rough, in fact one could devise some more elegant map but the discrepancy will be the same.

Theorem 1. Let C_N and C_M be respectively the N-chain and the M-chain, equipped with the same counting measure. Then:

$$d_{GH}(C_N, C_M) = |M - N| \tag{2.5}$$

Proof. Let $M > N$ then it is straightforward to find the map $f : C_N \to C_M$ with $\Delta_f = 0$. Considering the largest Lorentzian distances in the two causets, we realise they are the ones from the minimal to the maximal element in the chain[8]. The value is then $N - 1$ and $M - 1$ respectively.

Thus, for any choice of g,

$$\delta_g(y_0, y_{M-1}) = |M - 1 - d_{C_N}(g(y_0), g(y_{M-1}))| \geq M - N \qquad \square$$

Note that the distance is the difference in cardinality and not the fractional difference. Thus according to d_{GH}, a very long chain is as far from the chain with three more elements as the single element causet is far from the 4-chain. This might feel unsatisfactory, but it results from choosing the same counting measure on both causets. While this measure is interesting in the context of studying the space of causets in itself, we are not forced to choose it.

2.3 Chains with different measure

d_{GH} could be useful in studying coarse grainings of causets. In this case, we can adjust the weight on the counting measures according to the cardinality of the two spaces, so that they are as close as possible. Consider again the chains C_N and C_M, $N < M$, this time with measures:

$$\begin{cases} \mu_{C_N}(\{x_i\}) = 1 \ \ \forall x_i \\ \mu_{C_M}(\{y_i\}) = \alpha \ \ \forall y_i \end{cases}$$

[8]Maximal/minimal elements are elements with no successors/predecessors.

Depending on the value of α, $\Delta_f = 0$ might not be achievable. If $\alpha \neq 1$, embedding C_N in C_M just like we did earlier will result in $\Delta_f = |1-\alpha|(N-1)$. However, consider the map $f(x_i) = y_{ai}$ for some integer a.

$$\delta_f(x_i, x_j) = |1-\alpha a|(j-i)$$

For consistency, $a \leq \frac{M-1}{N-1}$. Let's consider for simplicity the case in which $\frac{M-1}{N-1}$ is indeed an integer. We set:

$$\begin{cases} a = \dfrac{M-1}{N-1} \\ \alpha = \dfrac{1}{a} \end{cases}$$

so that f maps the short chain evenly on the long one, with zero discrepancy[9].

We have to now consider g. Earlier, the longest chain difference represented a lower bound for Δ_g. Now:

$$\begin{cases} d_{C_N}(x_0, x_{N-1}) = N-1 \\ d_{C_M}(y_0, y_{M-1}) = \alpha(M-1) \end{cases}$$

So that $\alpha = \frac{N-1}{M-1}$ makes the longest chain difference null. There are other limitations to the closeness to consider. In particular, the fact that g is necessarily surjective. For two elements $y_i \prec y_{i'}$ that are mapped to the same element in C_N:

$$\delta_g(y_i, y_{i'}) = \alpha(i'-i) \geq \alpha$$

It can be arranged so that a maximum of b elements are mapped to the same point, where b is the first integer $b \geq \frac{M}{N}$, and also:

$$y_i \prec y_j \implies g(y_i) \preceq g(y_j)$$

So that g respects the causal structure. The discrepancy is then $\Delta_g = (b-1)\alpha$. Observe

$$1 > (b-1)\alpha \geq \frac{M-N}{N}\frac{N-1}{M-1}$$

Thus, for the choice of maps described, the two chains are more than 1-close, implying

$$d_{GH}(C_N, C_M) < 1 \tag{2.6}$$

when $\frac{M-1}{N-1}$ is an integer, and for this choice of measures.

A couple of remarks. This is just an upper bound, we did not prove this is in fact the best achievable result. Also, see that, for long chains,

$$\frac{M-N}{N}\frac{N-1}{M-1} \simeq 1 - \frac{N}{M}$$

If one would use d_{GH} to compare causets in the context of coarse graining, this would mean the longer the coarse grained chain is, the closer it potentially is. Finally, the case when $\frac{M-1}{N-1}$ is not an integer, one has to consider other maps. Possibly, there is a choice of maps with low discrepancy, but I did not have time to look for it.

We have seen that two chains are closest when their total measure is identical, and that d_{GH} can potentially be smaller if N is relatively large. Arguably, d_{GH} could be used to judge the effectiveness of a coarse graining. However this is just the simplest case of two chains. Causets are generally extremely more complex, and a much more detailed study is required.

[9] If $\frac{M-1}{N-1}$ were not an integer, one would have to take its integer part, and then $\Delta_f = (a\alpha - 1)(N-1)$.

2.4 Antichain

Another very simple causet is the antichain, in which no element is commensurable using the partial order, dust. If A_N is a N-antichain, and X is any other causal measure, then, no matter what maps (f, g) are chosen,

$$\Delta_g = \sup\{d_X(x_i, x_j) | x_i, x_j \in X\}$$

because $d(a_i, a_j) = 0$ for all $a \in A_N$. Invariably, then:

$$d_{GH}(A_N, X) = \sup\{d_X(x_i, x_j) | x_i, x_j \in X\} \tag{2.7}$$

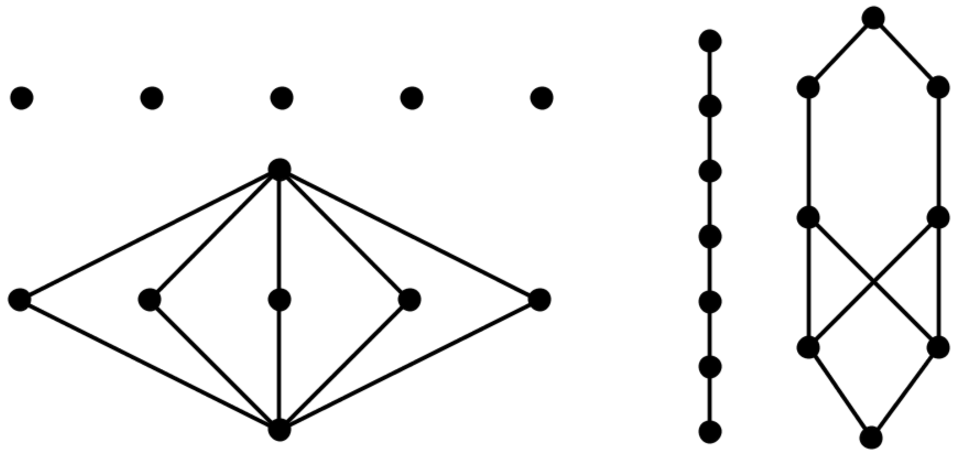

Figure 10: The 5-antichain, and four causets which, equipped with the same measure, are all the same distance d_{GH} away from it. Note the causet that has the antichain as a subcauset is no closer to the antichain than the chain is.

Thus, when comparing any causet to an antichain using d_{GH}, only the largest measure Alexandrov interval matters. Applied to the causets, these results prove d_{GH} does not provide a clear notion of being non-antichain like. For example, the causets in Figure 10 all are the same d_{GH} distance from an antichain. It does not select a particular configuration.

2.5 A manifoldlike causet

We consider here the important issue of using d_{GH} to quantify the embeddability of a causet into a given manifold. We would like a low d_{GH} distance to mean that a causet closely resembles a manifold. Here we prove an encouraging result in the case of a 1D manifold.

Theorem 2. For the N-chain C_N and the timelike line l of duration τ, under an appropriate choice of measures,

$$d_{GH}(C_N, l) = \frac{\tau}{N} \tag{2.8}$$

A constructive proof is presented[10].

Proof. Let the measure on the chain be such that $\mu(\{c\}) = \alpha$ for all $c \in C_N$, where α is a positive number to be specified. It follows that when $i < j$:

$$d(c_i, c_j) = \alpha(j - i)$$

[10]In the hope it is of any interest to reader wanting to tackle more complicated problems.

Let the timelike line l of duration τ be the set of reals $[0, \tau]$, such that for all points:

$$x_1 < x_2 \iff x_1 \prec x_2$$

The measures on the open sets are integrals on the line. To find a map $f : C_N \to l$ such that $\Delta_f = 0$, let

$$f(c_n) = \frac{\tau}{M} n$$

for all $n = 0, ..., N-1$, and where $M \geq N-1$ is a real number. f maps the points from the chain to successive regular intervals of l of duration $\frac{N}{M}\tau$. This is chosen to reflect the regularity of the two spaces[11]. For $0 \leq n < m \leq N-1$ we have:

$$\delta_f(c_n, c_m) = \left|\alpha(m-n) - \frac{\tau}{M}(m-n)\right|$$

and 0 otherwise. Thus:

$$\Delta_f = (N-1)\left|\alpha - \frac{\tau}{M}\right|$$

and we see then that it vanishes for the choice:

$$M = \frac{\tau}{\alpha}$$

Which requires $\tau \geq \alpha(N-1)$, because $M \geq N-1$.

Let's look now at the map $g : l \to C_N$. Let g be mapping successive intervals of same length to successive elements of the chain: $g(x) = c_n$ such that:

$$\begin{cases} n\dfrac{\tau}{N} \leq x \leq (n+1)\dfrac{\tau}{N} & \text{if } 0 \leq x < \tau \\ g(\tau) = c_{N-1} \end{cases} \tag{2.9}$$

Then,

$$\begin{cases} \delta_g(x_1, x_2) = 0 & \text{if } x_1 \geq x_2 \\ \delta_g(x_1, x_2) = |(x_2 - x_1) - \tau(g(x_2) - g(x_1))| & \text{if } x_1 < x_2 \end{cases} \tag{2.10}$$

For the second quantity, there are two cases to consider:

- *Same image:*

$$g(x_1) = g(x_2) \iff x_{1,2} = n\frac{\tau}{N} + \varepsilon_{1,2}$$

  where $0 \leq \varepsilon_1 \leq \varepsilon_2 \leq \frac{\tau}{N}$ (the last equality can only hold when $n = N-1$). It follows:

$$\delta_g(x_1, x_2) = |\varepsilon_2 - \varepsilon_1| \leq \frac{\tau}{N}$$

- *Different image:*

$$g(x_1) \prec g(x_2) \iff x_{1,2} = n_{1,2}\frac{\tau}{N} + \varepsilon_{1,2}$$

  with $0 \leq n_1 < n_2 \leq N-1$ and $0 \leq \varepsilon_1 < \frac{\tau}{N}$ and $0 \leq \varepsilon_2 \leq \frac{\tau}{N}$ (again the last equality can only hold when $n_2 = (N-1)$). Then:

$$\delta_g(x_1, x_2) = \left|(n_2 - n_1)\left(\frac{\tau}{N} - \alpha\right) + \varepsilon_2 - \varepsilon_1\right|$$

  Since $|\varepsilon_2 - \varepsilon_1| \leq \frac{\tau}{N}$ and $(n_2 - n_1) < N-1$,

$$\delta_g(x_1, x_2) \leq (N-1)\left|\frac{\tau}{N} - \alpha\right| + \frac{\tau}{N}$$

[11] The choice of mapping from the bottom is arbitrary. Starting to map from the top down, or to map in the middle of the line is equivalent, as long as the intervals are regular, and gives the same result. Only differences matter.

We set $\alpha = \frac{\tau}{N}$, then:

$$\Delta_g = \sup\{\delta_g(x_1, x_2)\} = \frac{\tau}{N}$$

Note that, otherwise, Δ_g would diverge with N: it minimises discrepancy to assign to each causet element the same measure as the set that is mapped to it. Since the requirements $\tau \geq \alpha(N-1)$ and $\alpha = \frac{\tau}{N}$ are compatible, we conclude that:

$$d_{GH}(C_N, l) \leq \frac{\tau}{N}$$

Thus, a line and a chain necessarily get closer as N gets larger *at least* as fast as N^{-1}. This is already a good result. To prove the equality it remains to show that modifications to these maps imply a larger discrepancy. Keep in mind that the discrepancies Δ of the functions correspond to the worst pair of points and images.

Thus, consider an alternative map g' such that $g'(x) = g(x)$ for all points on l except for a particular point y, which is mapped to a point $c_{k'} = g'(y) \prec g(y) = c_k$. This mapping interferes with the causal structure. The discrepancies δ_g are the same, except those with y. For example, consider an element $x = n\frac{\tau}{N} + \varepsilon$ such that $x \prec y$ but $g'(y) \prec g'(x) = g(x) = c_n$.

$$\begin{cases} \delta_g(x, y) = |y - x| \\ \delta_g(y, x) = \left|0 - (n - k')\frac{\tau}{N}\right| \end{cases} \tag{2.11}$$

Notice that, at best, these two quantities are less than $\frac{\tau}{N}$, but might be larger, depending on the choice of y. And introducing more points like y, will not make it better. Thus a map that does not respect the causal order cannot be better than g.

Any map g'' that respects the causal structure, will satisfy:

$$x_1 < x_2 \implies g''(x_1) \preceq g''(x_2)$$

which means that, just like g, it maps successive intervals of the line to successive elements on the causet. The freedom left is then on choosing the length of the intervals or the number of elements in C_N to which map to. Both modifications imply that at least one interval is larger than $\frac{\tau}{N}$, which makes $\Delta_{g''} > \Delta_g$, since these are in fact the size of the largest interval. Consider for example making the first longer and the second shorter by an amount λ:

$$\begin{cases} g''(x) = c_0 & \text{for } 0 \leq x < \frac{\tau}{N} + \lambda \\ g''(x) = c_1 & \text{for } \frac{\tau}{N} + \lambda \leq x < 2\frac{\tau}{N} \\ g''(x) = g(x) & \text{for } 2\frac{\tau}{N} \leq x << \tau \end{cases} \tag{2.12}$$

Let

$$\begin{cases} x_1 = 0 + \varepsilon_1 & \text{where } 0 \leq \varepsilon_1 < \frac{\tau}{N} + \lambda \\ x_2 = n\frac{\tau}{N} + \varepsilon_2 & \text{where } 0 \leq \varepsilon_2 < \frac{\tau}{N} \end{cases} \tag{2.13}$$

Then

$$\delta(x_1, x_2) = |\varepsilon_2 - \varepsilon_1| \leq \frac{\tau}{N} + \lambda$$

and we see $\Delta_{g''} > \Delta_g$, the size of the largest interval. Having exhausted the possibilities, the result is proven. □

The result is consistent with intuition: the longer the chain, the closer to the line. Also the maps that minimise discrepancy are the most intuitive ones. In this sense d_{GH} gives a very good result in this 1D case.

It is worth mentioning that in reality we expect the number of causet elements to correspond to a finite volume, and thus that it should not be possible to fit an infinite number of elements on a finite line, in contradiction to this result. In fact, nowhere in the definition of d_{GH} was something to prevent "over-packing". This does not make d_{GH} irrelevant, however. Embeddings are only interesting when considering large scales, and coarse grainings. At the Planck scale, spacetime is not manifold-like and the causet is not expected to embed.

2.6 Non-distinguishing causets

In the paper, a point is made about the positive definiteness of d_{GH} in non-distinguishing causal measure spaces. Here is an example where d_{GH} fails to be positive definite:

Example 1. Consider the 3-chain and the Y poset. They look different to the eye, one is a totally ordered universe, the other is a branching universe. However, by trying the mappings in Figure 11, we see $d_{GH}(C_3, Y) = 0$. The addition of one element to the past of both causets will not change the result.

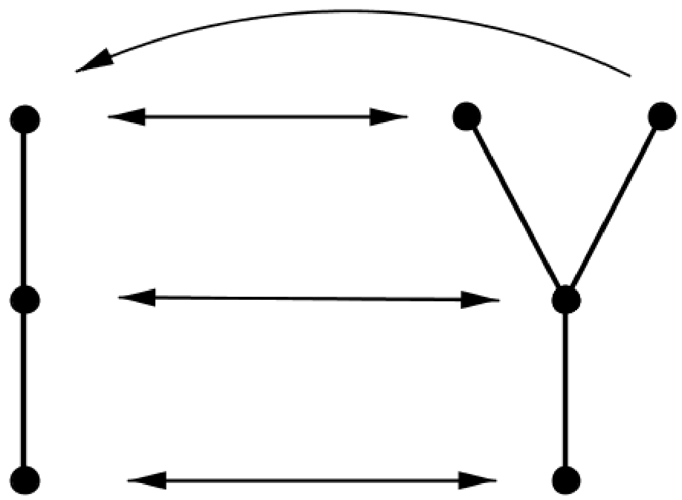

Figure 11: The best possible pair of maps from the 3-chain to the Y causet.

Causet elements do not exist on their own: they are part of a network and their existence is based on their relations with the other elements. Therefore elements with same past and future cannot be distinguished in any way, and might pose a technical problem, which might turn into an opportunity. Let's discuss this, and learn more about d_{GH}.

Definition 2.5. A causal measure space X is distinguishing if, for all $x_1, x_2 \in X$:

$$J^{\pm}(x_1) = J^{\pm}(x_2) \iff x_1 = x_2 \tag{2.14}$$

Definition 2.6. In a non-distinguishing causet, elements having the same past and future are termed non-Hegelian[12]. This is an equivalence relation '$\sim$':

$$x_1 \sim x_2 \iff J^{\pm}(x_1) = J^{\pm}(x_2) \tag{2.15}$$

It can be desirable to remove non-Hegelian elements. This is done by taking the quotient of the causet under the equivalence relation.

Definition 2.7. If X is a poset, denote by $\tilde{X}$ its Hegelianised version:

$$\tilde{X} = \{[x], \forall x \in X\} \tag{2.16}$$

where the causal structure is inherited from X.

We see that in the previous example, the 3-chain is the non-Hegelian version of the Y. There is a general result.

[12] It has been argued (F. Dowker, project meeting, January 2013) that the indistinguishability of posets elements can be attributed to their lack of haecceity, compared to other possible philosophical properties. The term stuck. It probably first appeared in [12].

Theorem 3. Let X be a causet having non-Hegelian maxima or minima, and $\tilde{X}$ its Hegelianised version. Then:

$$d_{GH}(X, \tilde{X}) = 0 \tag{2.17}$$

Proof. Let X be such a causet. Then, for any element $x \in X$:

$$\text{card } [x] \neq 1 \iff \big(J^+(x) = \emptyset\big) \vee \big(J^-(x) = \emptyset\big)$$

There are no non-Hegelian elements in any of the Alexandrov intervals of X thus

$$d_X(x_i, x_j) = d_{\tilde{X}}([x_i], [x_j])$$

Mapping all elements in X to their equivalence class in $\tilde{X}$ and conversely mapping all equivalence classes to one of their elements results in no discrepancy at all. □

Therefore, d_{GH} is not influenced by this future or past branching. Let's consider an example where non-Hegelian elements appear inside the causet.

Example 2. A simple causet and its Hegelianised version are shown in Figure 12, along with the best pair of maps. The value of d_{GH} is 1, and this is due to the discrepancy in mapping pairs of points that contain the bubble in their Alexandrov interval. Note that adding identical futures or pasts to both causets will not increase the distance, because, as pointed out earlier, d_{GH} only counts the worst discrepancy. Additionally, adding another element in the bubble, and performing an analogue mapping will show that the distance increases by 1.

It looks like d_{GH} counts the number of elements in the bubble.

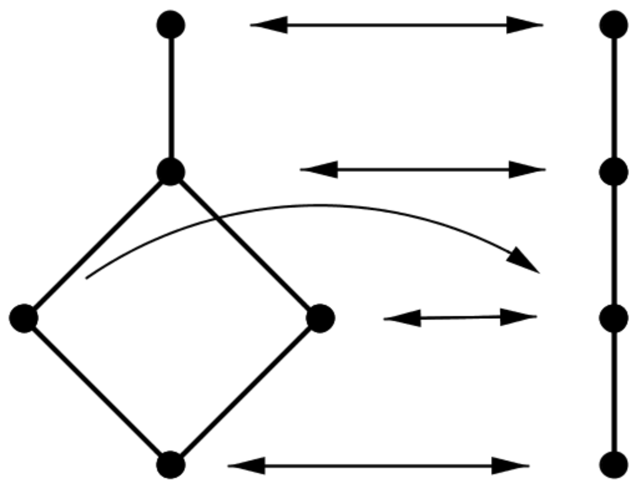

Figure 12: An example of a causet with non-Hegelian bubble and its Hegelianised version. Adding isomorphic successors and predecessors will not increase the distance.

We make the following conjecture:

Conjecture 1. Let X be a poset with exactly one non-Hegelian bubble containing n elements. Then

$$d_{GH}(X, \tilde{X}) = n - 1 \tag{2.18}$$

To motivate it more, consider the causets X and $\tilde{X}$. f maps all the elements from X to their equivalence class in $\tilde{X}$, and g maps each of the equivalence classes to one of their elements. Label y_i, $i = 0, 1, ..., n-1$, the elements in the non-Hegelian bubble. They have the following properties:

$$\begin{cases} [y_i] = [y_j] \equiv Y \\ y_i \not\prec y_j \\ J\pm(y_i) = J\pm(y_j) & \forall i, j = 0, 1, ..., n-1 \end{cases}$$

Thus, for any $x \notin Y$:

$$\delta_f(x, y_i) = 0 \qquad \delta_g([x], Y) = 0$$
$$\delta_f(y_i, y_j) = 0 \qquad \delta_g(Y, [x] = 0$$

because the Alexandrov sets involved are equinumerous.

Also, for elements x_i, x_j such that the non-Hegelian is in their Alexandrov intervals have

$$\delta_f(x_i, x_j) = 0 \qquad \delta_g([x_i], [x_j]) = 0$$

However, if the bubble is in indeed in their Alexandrov intervals then

$$d_X(x_i, x_j) = d_{\tilde{X}}([x_i], [x_j]) + (n-1)$$

so that the discrepancies are

$$\delta_f(x_i, x_j) = \delta_g([x_i], [x_j]) = n - 1$$

Thus, for this choice of maps, X and $\tilde{X}$ are $(n-1)$-close. The conjecture holds in some particular case, for example:

Theorem 4. If $\tilde{X}$ is a chain, then $d_{GH}(X, \tilde{X}) = n - 1$.

Proof. In the case where $\tilde{X}$ is a chain, the largest Lorentzian distance in both sets is from the minimal element to the maximal element, and they differ exactly by $n-1$. This is the lower bound for d_{GH} hence the theorem. □

In the general case, however, d_{GH} could in principle be less than $n-1$. A map fairly different from the one just considered might exist having lower discrepancy. A general proof has to be found. There are two similar conjectures.

Conjecture 2. If there are more than one bubble in the causet, and they are not commensurable, then d_{GH} will count the elements in the largest bubble.

Conjecture 3. If the are more than one bubble and they are related, d_{GH} would equal the sum of the bubble elements minus two.

They both hold for various particular cases, but might in principle not hold in general.

A reason to consider these conjectures is that non-Hegelian elements are particular and pose questions. Are they just redundant and need to be eliminated from the theory [12]? Or are they interesting, opening up possibilities in sequential growth models [5]? Or even being used as scalar fields, and should be studied? It would be helpful to find and quantify them. Additionally, it would be possible to make d_{GH} distinguish when having a causet X with non-Hegelian maxima or minima, such that $d_{GH}(X, \tilde{X}) = 0$. Create a new one X' by adding two elements, one preceding all elements, and one succeeding them all. Then considering $d_{GH}(X', \tilde{X}')$ one would find a non-zero value corresponding to the largest newly formed non-Hegelian bubble.

The answer probably needs to be sought in the more general question: when is it sound to compare subcausets? Intuition would say that two roughly similar causets will have a low d_{GH} corresponding to the bits that are most different. But how is it possible to rule out the existence of a maverick map, which outperforms the most natural one, the one preserving most of the causal structure? This is an important question to ask about the operational aspect of d_{GH}, because if, for some quirk, two different spaces are very close, then d_{GH} would prove to be less useful than what it promises.

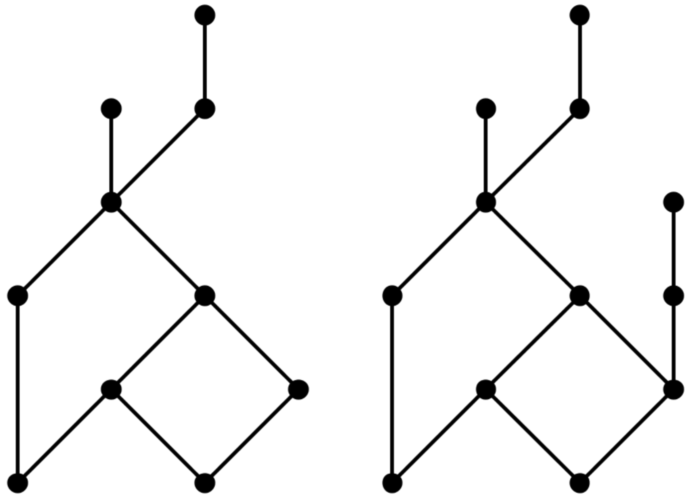

Figure 13: Two very similar causets. The question is, how can one be sure $d_{GH} = 2$ without trying all possible permutations?

2.7 Graph distance and d_{GH}

One final remark on the argument of studying the set of N-causets. It is possible to compare all distinguishing causets with a given number of elements and draw a distance graph using d_{GH}. This will generally look very different from the typical distance graph that is obtained by counting local moves, and thus offers new possibilities of insight.

One difference is the general length of the graph. The largest d_{GH} distance in the set of N-causets is $N-1$, and corresponds to the distance from the N-antichain to the N-chain, but not exclusively, as seen on Figure 10. In contrast, the maximum graph distance is ${}^{N}C_2$, and corresponds to the chain and antichain only. Also, the d_{GH} distance graph is more connected, and, at least for small N, has an interesting layered structure. A more detailed discussion can be found in the reports written by M. Al-Khalil [10] and S. Pallister [11], the other two students working on the project.

2.8 Recap Remarks

- d_{GH} only counts the worst discrepancy in the best possible map. There is no accumulation or averaging of discrepancies.
- When a causal measure space is isomorphic to a subset of another, then it can be mapped to it with zero discrepancy, provided they are equipped with the same measure function. The value of d_{GH} is then the discrepancy of the other map.
- A lower bound for d_{GH} is always the difference of the largest Lorentzian distances in the two spaces.
- Comparing causets: The N-chain and M-chain, where their measure is such that each single element measures 1, are $|M-N|$ apart according to d_{GH}.
- Coarse grainings: If we are allowed to chose the measure of the chains, they can be closer than 1.
- Embeddings: The distance between the chain and a finite causal line is the length of the line divided by the number of elements in the chain.
- d_{GH} fails to be positive definite in the context of non distinguishing causets. Some regularities are found, but more study is needed here.

Conclusion

We have explicitly seen just how much information is encoded in the causal structure. This as been a personal humbling eye-opener in the context of understanding physical phenomena. More practically, the introduction of causal measure spaces and the function d_{GH} offer an interesting opportunity. The distance function has properties relating to embeddings and coarse grainings that, if they carry on to higher dimensions and larger causets, would make it an interesting candidate for a distance function. A difficulty however, as in studying causal sets in general, stems from the discreteness, which makes calculations non-analytical.

I will take the occasion to thank Fay Dowker for having been a committed, inspiring and generous supervisor and Jack Jelfs for having been such a patient companion and intrepid causalnaut; they have been fundamental in the development and enjoyment of the project and I wish them a rich future. Also Majid Al-Khalil and Sam Pallister have been good collaborators, always available and producing a stream of interesting work on causets.

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Appendices

A From Lorentzian manifolds to Causal Sets

We spent part of the project time numerically exploring the properties of causets extracted from a background spacetime through a stochastic process called *sprinkling*. In this section we briefly summarise[13] the results we reproduced. A much more detailed account, including explanation of the algorithms, can be found in the report my project partner Jelfs has written [13]. The code we wrote and used is our work, fun to use and available online *here* [14].

A.1 Sprinklings

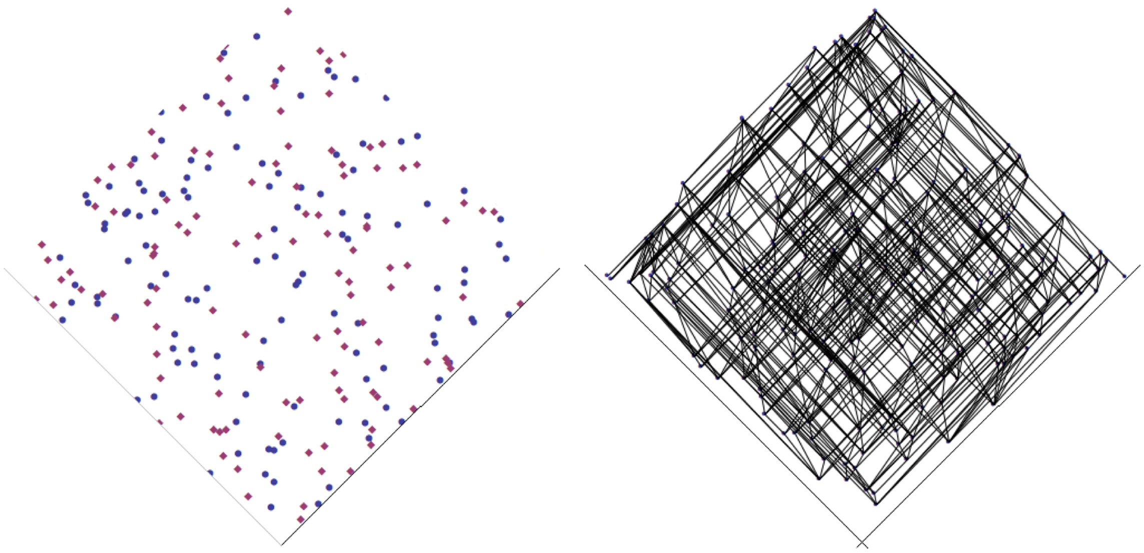

Figure 14: Two sprinklings on a portion of 1+1 dimensional Minkowski spacetime. The first shows the selected points, and the same points Lorentz boosted. The second shows explicitly the links between the selected points.

Sprinklings are made by selecting random points of the manifold through a Poisson process, and endowing them with the partial order of the manifold. The result is a causet which can perfectly be embedded[14] in the original manifold. The Poisson process is of particular interest because it produces a Lorentz invariant result. A boosted Poisson distribution is a Poisson distribution [4] (see Figure 14).

We wrote code in *Mathematica* to create causets from sprinklings in 1+1, 2+1, 3+1 dimensional Minkowski spacetime, as well as in a 1+1 dimensional cylinder and trousers[15] space times.

A.2 Dimension Estimation

We used the Myrheim-Meyer dimension estimator [4] to reproduce the results in Reid 2003 [15]. It works by selecting pairs of points and counting the number of links contained in their Alexandrov intervals. The dimension estimator was used in Minkowski spacetime and worked correctly. We tested it by selecting intervals of fixed size and then averaging the results. We tried various interval sizes, and the efficiency observed was as predicted by Sorkin [4].

[13] To give more importance to the original work produced during the rest of the project.

[14] Meaning there exists a map from causet to manifold that preserves the causal structure. It would be interesting to study the value of d_{GH} in this case. A numerical upper bound could be found quite easily.

[15] A cylinder spacetime that splits in two cylinders at a given point.

The cylinder spacetime is locally 2 dimensional, but looks more 1 dimensional at scales because if two points are separated in time by more than half the circumference, they are necessarily connected. This was reflected in the output of the dimension estimator for various interval sizes and sprinkling densities. Larger intervals give values closer to 1, smaller intervals closer to 2. Conversely, if the sprinkling density is low, the estimator's output is closer to 1.

A.3 Exposing topology

We also wrote code that selects an antichain using exclusively the partial order and "grows" it by adding only elements directly linked to it. The process is repeated and results in subsets that are discrete analogues of thickened spatial hyperplanes (Figure 15). When *Mathematica* displays causet as a directed graph, the hole is evident (Figure 16). The same was done for the trousers and resulted in either a causet with a single hole, or the disjoint union of two causets with a single hole, depending on the position of the antichain. This shows that the causal structure encodes the topology too, although no more detailed work was carried. Much more information on recovering topology can be found in [16].

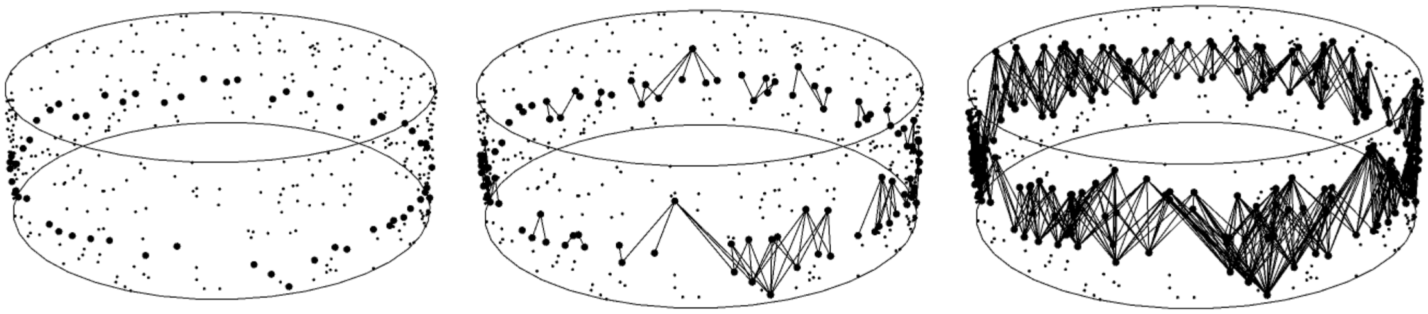

Figure 15: Sprinkling of 400 points on a cylinder spacetime. In the images, an antichain selected with our code is highlighted, then when it has been grown one step, then three steps.

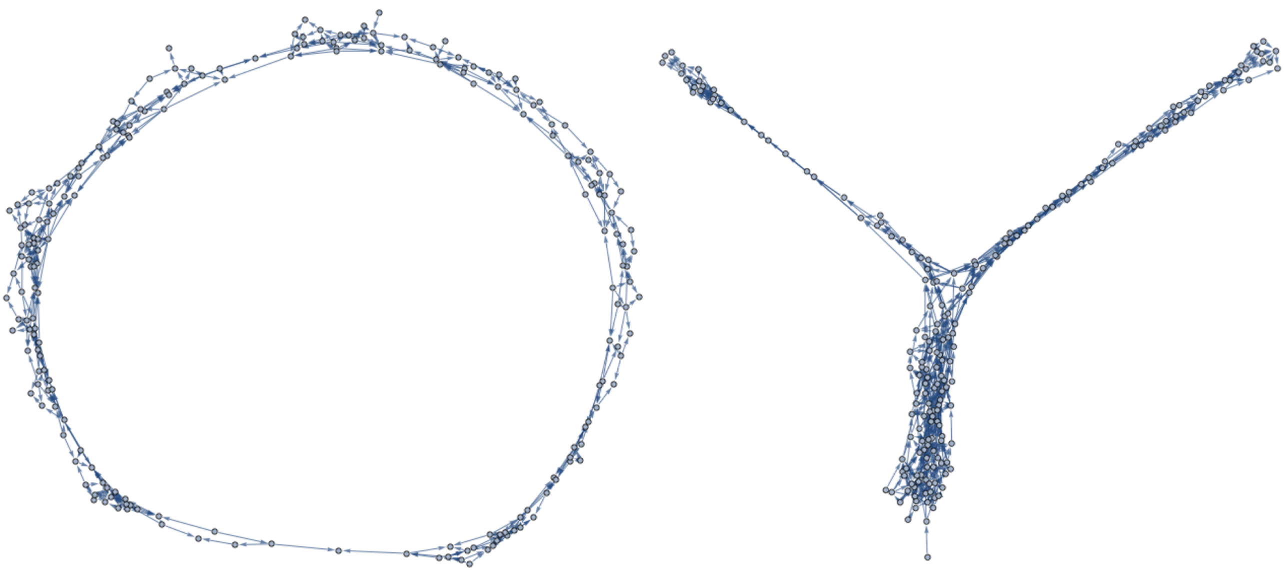

Figure 16: *Mathematica* output. The first image is the directed graph of the third subcauset in Figure 15. The Second image is a directed graph of a causet sprinkled on a trouser spacetime. In both the arrows are too small to see, but the general idea of, respectively, a hole and causal independent pieces is evoked.

B Breakdown of Original Work

Most of the project time was spent working individually, mainly working in pairs with Jack while developing the *Mathematica* code. We met our supervisor weekly and compared our results and shared ideas with the rest od the group.

- Section 1: This is original work and does not appear in the literature, although the proof programme - the order in which to find the causal definitions - was introduced to us by Prof Dowker and already mentioned by Sorkin[4].
- Section 2: Most of 2.1 was summarised from [9]. The rest of the section is independent work, including all derivations and remarks, and does not appear in the literature.
- Appendix A: We mainly reproduced work in the literature. The algorithms to extract the causal structure of the sprinkled causet was work Jack and I, except in the case of 1+1 Minkowski spacetime, which was courtesy of Prof Dowker. All code used was original code on *Mathematica*.