9 Derivatives, Part IIa (Differentiation)

9.1 Basic proofs

We now prove theorems that make differentiation of a large class of functions easy.

Theorem 1. If $f(x) = c$ then $f'(a) = 0$ for all a.

Intuitively derivatives measure the rate of change. A constant function doesn't change, thus the derivative is zero.

Proof: we already proved this in the previous chapter.

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Theorem 2. If $f(x) = x$ then $f'(a) = 1$ for all a.

Intuitively $f(x)$ grows at exactly the same rate as x, thus the derivative is 1.

Proof:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{a+h-a}{h} = 1$$

—

Theorem 3. If f, g are differentiable at a, then $(f+g)'(a) = f'(a) + g'(a)$.

Examples:

- You have two functions, each modeling growth of some bank account. You want to understand the rate of growth of both accounts.
- You have two different assembly lines producing the same product. $c_1(x)$ and $c_2(x)$ model the cost of producing x units on each assembly line. You want to understand total cost changes as production across both assembly lines increases.

Proof:

$$\begin{aligned}(f+g)'(a) &= \lim_{h\to 0} \frac{(f+g)(a+h) - (f+g)(a)}{h} \\ &= \lim_{h\to 0} \frac{f(a+h) + g(a+h) - f(a) - g(a)}{h} \\ &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} + \lim_{h\to 0} \frac{g(a+h) - g(a)}{h} \\ &= f'(a) + g'(a)\end{aligned}$$

—

Theorem 3a. If $f_1, \ldots, f_n$ are differentiable at a, then:

$$f_1 + \ldots_+ f_n)'(a) = f_1'(a) + \ldots_+ f_n'(a)$$

Proof. This is a fairly straightforward proof by induction. Skipping it here as I've already spent enough time on this chapter.

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Theorem 4. If f, g are differentiable at a, then

$$(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a)$$

Examples:

- Let $r_1(t), r_2(t)$ model the length of each side of a rectangle over time. You want to understand the change in area at time t.

Proof:

$$\begin{aligned}
(f \cdot g)'(a) = &= \lim_{h \to 0} \frac{(f \cdot g)(a+h) - (f \cdot g)(a)}{h} \\
&= \lim_{h \to 0} \frac{f(a+h)g(a+h) - f(a)g(a)}{h} \\
&= \lim_{h \to 0} \frac{f(a+h)g(a+h) - f(a)g(a) + f(a+h)g(a) - f(a+h)g(a)}{h} \\
&= \lim_{h \to 0} \frac{f(a+h)(g(a+h) - g(a)) + g(a)(f(a+h) - f(a))}{h} \\
&= \lim_{h \to 0} \left(f(a+h) \frac{g(a+h) - g(a)}{h} + g(a) \frac{f(a+h) - f(a)}{h} \right) \\
&= \lim_{h \to 0} f(a+h) \cdot \lim_{h \to 0} \frac{g(a+h) - g(a)}{h} + \lim_{h \to 0} g(a) \cdot \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \\
&= \lim_{h \to 0} f(a+h) \cdot g'(a) + g(a) \cdot f'(a)
\end{aligned}$$

Recall from 7.2 that if f is differentiable at a, then $\lim_{h \to 0} f(a+h) = f(a)$. Thus

$$(f \cdot g)'(a) = f(a) \cdot g'(a) + g(a) \cdot f'(a)$$

—

Theorem 4a. If $f_1, \ldots, f_n$ are differentiable at a, then:

$$f_1 \cdot \ldots \cdot f_n)'(a) = \sum_{i=1}^{n} f_1(a) \cdot f_i'(a) \cdot f_n(a)$$

Proof. This is a fairly straightforward proof by induction. Skipping it here as I've already spent enough time on this chapter.

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Theorem 5. If $g(x) = cf(x)$ then $g'(a) = c \cdot f'(a)$.

Examples:

- Let h be a height of a rectangle that's constant, and let $b(t)$ model the length of the base of a rectangle over time. You want to understand the change in area at time t.

Proof: Let $h(x) = c$ so $g = h \cdot f$. Then by theorem 4:

$$\begin{aligned} g'(x) &= h'(x)f(x) + f'(x)g(x) \\ &= 0 \cdot f(x) + cf'(x) \\ &= cf'(x) \end{aligned}$$

—

Theorems 1-5 imply:

$$(-f)'(a) = (-1 \cdot f)'(a) = -f'(a)$$

and

$$(f - g)'(a) = (f + (-g))'(a) = f'(a) + (-g)'(a) = f'(a) - g'(a)$$

—

Theorem 6. If $f(x) = x^n$ for $n \in \mathcal{N}$, then $f'(a) = na^{n-1}$ for all a.

Examples:

- Let $s(t)$ model the length of the side of a cube over time. You want to understand the change in volume at time t.

Proof. We prove this by induction. For $n = 1$, $f'(a) = 1$ by theorem 2.

Assume if $f(x) = x^n$ then $f'(a) = na^{n-1}$ for all a.

Let $I(x) = x$ and let $g(x) = x^{n+1} = xx^n$. Then $g(x) = I(x) \cdot f(x)$, i.e. $g = I \cdot f$. By theorem 4:

$$\begin{aligned} g'(a) &= (I \cdot f)'(a) \\ &= I'(a)f(a) + I(a)f'(a) \\ &= 1 \cdot a^n + a \cdot na^{n-1} \\ &= a^n + na^n \\ &= a^n(1 + n) \\ &= (n + 1)a^n \end{aligned}$$

—

Theorem 6b. If $f(x) = x^n$ for $n < 0$, then $f'(a) = na^{n-1}$ for all a. (In other words, we extend theorem 6 to negative exponents.)

Proof. We use theorem 7 below (putting 6b here for learning convenience).

$$\begin{aligned} f'(a) &= \left(\frac{1}{a^{-n}}\right)' \\ &= \frac{nx^{-n-1}}{x^{-2n}} \\ &= nx^{n-1} \end{aligned}$$

—

Theorem 7. If g is differentiable at a and $g(a) \neq 0$, then

$$\left(\frac{1}{g}\right)'(a) = \frac{-g'(a)}{[g(a)]^2}$$

Examples:

- Let $i(d) = \frac{1}{d^2}$ model the intensity of light, which is inversely proportional to the square of the distance from the source. You want to know how intensity changes with distance.

Proof. We will prove this by using the derivative definition. However, we must first show $\left(\frac{1}{g}\right)(a+h)$ is defined for sufficiently small h. This is easy.

Since g is differentiable at a it is continuous at a. Thus by nonzero neighborhood lemma (see 4.1) there exists $\delta > 0$ such that $|h| < \delta$ implies $g(a+h) \neq 0$ for all h. Thus $\left(\frac{1}{g}\right)(a+h)$ is defined for sufficiently small h.

We are now ready to prove the core of the theorem.

$$\begin{aligned} \lim_{h \to 0} \frac{\left(\frac{1}{g}\right)(a+h) - \left(\frac{1}{g}\right)(a)}{h} &= \lim_{h \to 0} \left(\frac{1}{g(a+h)} - \frac{1}{g(a)}\right)/h \\ &= \lim_{h \to 0} \left(\frac{g(a) - g(a+h)}{g(a) \cdot g(a+h)}\right)/h \\ &= \lim_{h \to 0} \frac{g(a) - g(a+h)}{h \cdot g(a) \cdot g(a+h)} \\ &= \lim_{h \to 0} \frac{-[g(a+h) - g(a)]}{h} \cdot \frac{1}{g(a) \cdot g(a+h)} \\ &= \lim_{h \to 0} \frac{-[g(a+h) - g(a)]}{h} \cdot \lim_{h \to 0} \frac{1}{g(a) \cdot g(a+h)} \end{aligned}$$

Recall from 7.2 that if f is differentiable at a, then $\lim_{h\to 0} f(a+h) = f(a)$. Thus:

$$\lim_{h\to 0} \frac{-[g(a+h)-g(a)]}{h} \cdot \lim_{h\to 0} \frac{1}{g(a)\cdot g(a+h)} = -g'(a)\cdot \frac{1}{[g(a)]^2}$$

as desired.

—

Theorem 8. If f, g are differentiable at a and $g(a) \neq 0$, then

$$\left(\frac{f}{g}\right)'(a) = \frac{g(a)\cdot f'(a) - f(a)\cdot g'(a)}{[g(a)]^2}$$

Examples:

- Let $e(t), s(t)$ model the number of engineers and sales people at a company over time. You want to understand the change in the ratio between the two.

Proof.

$$\begin{aligned}
\left(\frac{f}{g}\right)'(a) &= \left(f\cdot\frac{1}{g}\right)'(a) \\
&= f(a)\cdot\left(\frac{1}{g}\right)'(a) + f'(a)\cdot\left(\frac{1}{g}\right)(a) \\
&= \frac{-g'(a)\cdot f(a)}{[g(a)]^2} + \frac{f'(a)}{g(a)} \\
&= \frac{-g'(a)\cdot f(a)\cdot g(a) + f'(a)\cdot [g(a)]^2}{[g(a)]^3} \\
&= \frac{f'(a)\cdot g(a) - g'(a)\cdot f(a)}{[g(a)]^2}
\end{aligned}$$

9.2 Chain rule

The derivative of composed functions is considerably more complicated, and so deserves its own section. We'll prove this in two stages. First, we'll attempt a proof with a few false starts that will point us in the direction of a real proof. Once the direction becomes clear, we'll abandon our first draft and write a clean proof from scratch.

Theorem 9 (the chain rule). If g is differentiable at a, and f is differentiable at $g(a)$, then

$$(f\circ g)'(a) = f'(g(a))\cdot g'(a)$$

Examples:

- Let $a(t)$ model altitude of a rocket over time, and let $p(a)$ model air pressure at a particular altitude. You want to know how air pressure changes over time.

Proof, first draft.

As usual, we start with the definition of the derivative:

$$\begin{aligned}
(f \circ g)'(a) &= \lim_{h\to 0} \frac{(f \circ g)(a+h) - (f \circ g)(a)}{h} \\
&= \lim_{h\to 0} \frac{f(g(a+h)) - f(g(a))}{h} \\
&= \lim_{h\to 0} \left(\frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)} \cdot \frac{g(a+h) - g(a)}{h} \right) \\
&= \lim_{h\to 0} \frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)} \cdot \lim_{h\to 0} \frac{g(a+h) - g(a)}{h} \\
&= \left(\lim_{h\to 0} \frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)} \right) \cdot g'(a)
\end{aligned}$$

This is a bit of a false start as we now have two problems:

- To get $f'(g(a))$ in the first term, we need $\lim_{h\to 0} \frac{f(g(a)+h)-f(g(a))}{h}$, but instead we have $\lim_{h\to 0} \frac{f(g(a+h))-f(g(a))}{g(a+h)-g(a)}$.
- $g(a+h) - g(a)$ may be zero for $h \neq 0$, so the division may be illegal.

However it isn't a total waste. Our false start gives us an idea for how we may proceed– we'll replace $\frac{f(g(a+h))-f(g(a))}{g(a+h)-g(a)}$ with something better. What could be the replacement? Let's hypothesize existance of a function $\phi(h)$ with the following property (we will soon prove such a function exists):

$$\frac{f(g(a+h)) - f(g(a))}{h} = \phi(h) \cdot \frac{g(a+h) - g(a)}{h}$$

We can then rewrite our initial equations as follows:

$$\begin{aligned}
(f \circ g)'(a) &= \lim_{h\to 0} \frac{(f \circ g)(a+h) - (f \circ g)(a)}{h} \\
&= \lim_{h\to 0} \frac{f(g(a+h)) - f(g(a))}{h} \\
&= \lim_{h\to 0} \left(\phi(h) \cdot \frac{g(a+h) - g(a)}{h} \right) \\
&= \lim_{h\to 0} \phi(h) \cdot \lim_{h\to 0} \frac{g(a+h) - g(a)}{h} \\
&= \lim_{h\to 0} \phi(h) \cdot g'(a)
\end{aligned}$$

To get to $(f \circ g)'(a) = f'(g(a)) \cdot g'(a)$ we need $\phi(h)$ to possess one more property:

$$\lim_{h \to 0} \phi(h) = f'(g(a))$$

Given this additional property, we can now finish our reasoning:

$$(f \circ g)'(a) = \lim_{h \to 0} \phi(h) \cdot g'(a) = f'(g(a)) \cdot g'(a)$$

Thus proving the chain rule reduces to proving there exists a function $\phi(h)$ with the two properties above. For cleanliness, let's start a new proof from scratch and demonstrate the existance of such a function.

Proof.

Suppose there exists a function $\phi(h)$ with the following properties:

$$\frac{f(g(a+h)) - f(g(a))}{h} = \phi(h) \cdot \frac{g(a+h) - g(a)}{h} \quad (1)$$

$$\lim_{h \to 0} \phi(h) = f'(g(a)) \quad (2)$$

Then

$$\begin{aligned}
(f \circ g)'(a) &= \lim_{h \to 0} \frac{(f \circ g)(a+h) - (f \circ g)(a)}{h} \\
&= \lim_{h \to 0} \frac{f(g(a+h)) - f(g(a))}{h} \\
&= \lim_{h \to 0} \left(\phi(h) \cdot \frac{g(a+h) - g(a)}{h} \right) && \text{by property 1} \\
&= \lim_{h \to 0} \phi(h) \cdot \lim_{h \to 0} \frac{g(a+h) - g(a)}{h} \\
&= \lim_{h \to 0} f'(g(a)) \cdot g'(a) && \text{by property 2}
\end{aligned}$$

To complete the proof we must construct such a function and prove our construction has properties 1 and 2. We will do so now. Define ϕ as follows:

$$\phi(h) = \begin{cases} \frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)} & \text{if } g(a+h) - g(a) \neq 0 \\ f'(g(a)) & \text{if } g(a+h) - g(a) = 0 \end{cases}$$

We will prove properties 1 and 2 hold for ϕ.

Property 1 proof.
We now show $\frac{f(g(a+h)) - f(g(a))}{h} = \phi(h) \cdot \frac{g(a+h) - g(a)}{h}$. There are two cases: either $g(a+h) - g(a) \neq 0$ or $g(a+h) - g(a) = 0$. Suppose $g(a+h) - g(a) \neq 0$. Then

$$\begin{aligned}
\phi(h) \cdot \frac{g(a+h) - g(a)}{h} &= \frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)} \cdot \frac{g(a+h) - g(a)}{h} \\
&= \frac{f(g(a+h)) - f(g(a))}{h}
\end{aligned}$$

Alternatively, suppose $g(a+h) - g(a) = 0$. Then

$$\begin{aligned}\phi(h) \cdot \frac{g(a+h) - g(a)}{h} &= f'(g(a)) \cdot \frac{g(a+h) - g(a)}{h} \\ &= f'(g(a)) \cdot \frac{0}{h} \\ &= 0\end{aligned}$$

But $g(a+h) - g(a) = 0$ means $g(a+h) = g(a)$, and thus $\frac{f(g(a+h)) - f(g(a))}{h} = 0$. Thus in both cases property 1 holds, as desired.

Property 2 proof.
We now show $\lim_{h \to 0} \phi(h) = f'(g(a))$. Put differently:

- *Intuitively*, we're trying to show that when h is small, the top piece of ϕ piecewise definition approaches the bottom piece (which we chose to be $f'(g(a))$).
- Here is another way to frame it. Observe that $\phi(0) = f'(g(a))$. Thus showing $\lim_{h \to 0} \phi(h) = f'(g(a))$ is equivalent to showing $\lim_{h \to 0} \phi(h) = \phi(0)$, i.e. that ϕ is continuous at 0.
- Formally, we must show that given $\epsilon > 0$ there exists $\delta > 0$ such that $|h| < \delta$ implies $|\phi(h) - f'(g(a))| < \epsilon$.

So, let $\epsilon > 0$ be given.

Firstly, since f is differentiable at $g(a)$, by definition of the derivative we have:

$$f'(g(a)) = \lim_{k \to 0} \frac{f(g(a) + k) - f(g(a))}{k}$$

Inlining the limit defition, for all $\epsilon > 0$ there exists $\delta' > 0$ such that $0 < |k| < \delta'$ implies

$$\left| \frac{f(g(a) + k) - f(g(a))}{k} - f'(g(a)) \right| < \epsilon$$

Secondly, since g is differentiable at a, it continuous at a. Thus:

$$\lim_{h \to 0} g(a+h) = g(a)$$

Or put differently, there exists $\delta > 0$ such that $|h| < \delta$ implies:

$$|g(a+h) - g(a)| < \delta'$$

Finally, we now have everything we need to prove property 2. Consider any h with $|h| < \delta$.

- If $g(a+h) - g(a) = 0$ then $\phi(h) = f'(g(a))$ so $|\phi(h) - f'(g(a))| < \epsilon$.

- If $g(a+h) - g(a) \neq 0$ we can fix $k = g(a+h) - g(a)$ as both aren't 0 and are less than δ'. Thus we get:

$$\begin{aligned}
\epsilon &> \left| \frac{f(g(a)+k) - f(g(a))}{g(a+h) - g(a)} - f'(g(a)) \right| \\
&= \left| \frac{f(g(a)+g(a+h)-g(a)) - f(g(a))}{g(a+h) - g(a)} - f'(g(a)) \right| \\
&= \left| \frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)} - f'(g(a)) \right| \\
&= |\phi(h) - f'(g(a))|
\end{aligned}$$

I.e. $|\phi(h) - f'(g(a))| < \epsilon$ as desired.

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Theorem 9a. Let f_i be differentiable at $f_{i+1}(\ldots f_n(x) \ldots)$. Then:

$$(f_1 \circ \ldots \circ f_n)'(x) = \prod_{i=1}^{n} f_i' \left(f_{i+1}(\ldots f_n(x) \ldots) \right)$$

Proof. This is a fairly straightforward proof by induction. Skipping it here as I've already spent enough time on this chapter.

9.3 Derivatives of polynomials

We can easily find derivatives of polynomials using theorems 1-6. It turns out to be an interesting enough form that it's worth mentioning explicitly. Consider

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

Then:

$$f'(x) = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \ldots + 2 a_2 x + a_1$$

Continuing:

$$f''(x) = n(n-1) a_n x^{n-2} + (n-1)(n-2) a_{n-1} x^{n-3} + \ldots + 2 a_2$$

Repeatedly continuing this process we get:

$$f^{(n)}(x) = n! a_n$$

And of course for $m > n$ it's easy to see $f^{(m)} = 0$.

9.4 Differentiation practice

Spivak spends a lot of the chapter covering concrete differentiation examples. I work through these here. First, a summary of the nine differentiation theorems proved above:

1. If $f(x) = c$ then $f'(a) = 0$.
2. If $f(x) = x$ then $f'(a) = 1$.
3. $(f+g)'(a) = f'(a) + g'(a)$.
4. $(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a)$.
5. If $g(x) = cf(x)$ then $g'(a) = c \cdot f'(a)$.
6. If $f(x) = x^n$ for $n \in \mathcal{N}$, then $f'(a) = na^{n-1}$.
7. $\left(\frac{1}{g}\right)'(a) = \frac{-g'(a)}{[g(a)]^2}$.
8. $\left(\frac{f}{g}\right)'(a) = \frac{g(a) \cdot f'(a) - f(a) \cdot g'(a)}{[g(a)]^2}$.
9. $(f \circ g)'(a) = f'(g(a)) \cdot g'(a)$.

You also need to know two trig derivatives presented below without proof (proper proofs will show up in a later chapter when sin and cos are formally defined):

$$\sin'(a) = \cos a$$
$$\cos'(a) = -\sin a$$

We are now ready to practice example problems.

$$\begin{aligned}
f(x) = \frac{x^2-1}{x^2+1} &\implies f'(x) = \frac{(x^2+1)2x - (x^2-1)2x}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2} \\
f(x) = \frac{x}{x^2+1} &\implies f'(x) = \frac{1-x^2}{(x^2+1)^2} \\
f(x) = \frac{1}{x} = x^{-1} &\implies f'(x) = -\frac{1}{x^2} = (-1)x^{-2} \\
f(x) = x \sin x &\implies f'(x) = \sin x + x \cos x \\
&\implies f''(x) = 2\cos x - x \sin x \\
g(x) = \sin^2 x = \sin x \sin x &\implies g'(x) = 2 \sin x \cos x \\
&\implies g''(x) = 2\cos^2 x - 2\sin^2 x \\
h(x) = \cos^2 x = \cos x \cos x &\implies h'(x) = -2 \sin x \cos x \\
&\implies h''(x) = 2\sin^2 x - 2\cos^2 x
\end{aligned}$$

Note $g'(x) + h'(x) = 0$. This is something we could have guessed– $(g + h)(x) = \sin^2 x + \cos^2 x = 1$, thus by theorem 1, $(g + h)'(x) = 0$.

$$\begin{aligned} &f(x) = x^3 \sin x \cos x \\ &\implies f'(x) = 3x^2 \sin x \cos x + x^3 \cos^2 x - x^3 \sin^2 x \end{aligned}$$

The next set of examples uses the chain rule (where sometimes the product rule could be used instead). For example, $\sin^2 x$ could be interpreted either as $\sin x \sin x$, or as $s(\sin x)$ where $s(x) = x^2$.

$$\begin{aligned} f(x) = \sin x^2 &\implies f'(x) = \cos x^2 \cdot 2x \\ f(x) = \sin^2 x &\implies f'(x) = 2 \sin x \cdot \cos x \\ f(x) = \sin x^3 &\implies f'(x) = \cos x^3 \cdot 3x^2 \\ f(x) = \sin^3 x &\implies f'(x) = 3 \sin^2 x \cdot \cos x \\ f(x) = \sin \frac{1}{x} &\implies f'(x) = cos \frac{1}{x} \cdot \frac{-1}{x^2} \\ f(x) = \sin(\sin x) &\implies f'(x) = \cos(\sin x) \cdot \cos x \\ f(x) = \sin(x^3 + 3x^2) &\implies f'(x) = \cos(x^3 + 3x^2) \cdot (3x^2 + 6x) \\ f(x) = (x^3 + 3x^2)^{53} &\implies f'(x) = 53(x^3 + 3x^2)^{52} \cdot (3x^2 + 6x) \end{aligned}$$

We now consider a composition of three functions:

$$\begin{aligned} f(x) = \sin^2 x^2 = s \circ (\sin \circ s) &\implies f'(x) = 2 \sin x^2 \cdot \cos x^2 \cdot 2x \\ f(x) = \sin(\sin x^2) = \sin \circ (\sin \circ s) &\implies f'(x) = \cos(\sin x^2) \cdot \cos x^2 \cdot 2x \end{aligned}$$

And finally a composition of four functions:

$$\begin{aligned} &f(x) = \sin^2(\sin^2 x) = s \circ (\sin \circ (s \circ \sin)) \\ &\implies f'(x) = 2 \sin(\sin^2 x) \cdot \cos(\sin^2 x) \cdot 2 \sin x \cdot \cos x \\ &f(x) = \sin((\sin x^2)^2) = \sin \circ s \circ \sin \circ s \\ &\implies f'(x) = \cos((\sin x^2)^2) \cdot 2 \sin x^2 \cdot \cos x^2 \cdot 2x \\ &f(x) = \sin^2(\sin(\sin x)) = s \circ \sin \circ \sin \circ \sin \\ &\implies f'(x) = 2 \sin(\sin(\sin x)) \cdot \cos(\sin(\sin x)) \cdot \cos(\sin x) \cdot \cos x \end{aligned}$$

9.5 Sine polynomials

I don't think "sine polynomials" is a real name, but I needed a clever name for this section. Here we explore derivatives of functions of the form $x^k \sin \frac{1}{x}$.

Claim 1: Let

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Then f is not differentiable at 0.

Proof. Using derivative definition:

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{h\sin\frac{1}{h}-0}{h} = \lim_{h\to 0}\sin\frac{1}{h}$$

We saw in 8.3 that $\lim_{h\to 0}\sin\frac{1}{h}$ does not exist. Thus f is not differentiable at zero.

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Claim 2: Let

$$f(x) = \begin{cases} x^2\sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Then f is differentiable at 0.

Proof. Using derivative definition:

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{h^2\sin\frac{1}{h}-0}{h} = \lim_{h\to 0}h\sin\frac{1}{h} = 0$$

Thus $f'(0) = 0$.

—

Claim 3: Let

$$f(x) = \begin{cases} x^2\sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Then f' is not differentiable at 0.

Proof. Observe that:

$$f'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Observe that $\lim_{x\to 0}\cos\frac{1}{x}$ does not exist (for the same reason $\lim_{x\to 0}\sin\frac{1}{x}$ does not exist). Thus $\lim_{x\to 0}f'(x)$ does not exist. And thus f' is not continuous, let alone differentiable at 0.