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QISS

The arrow of time in operational formulations of quantum theory

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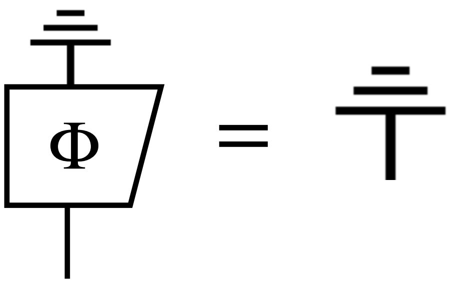
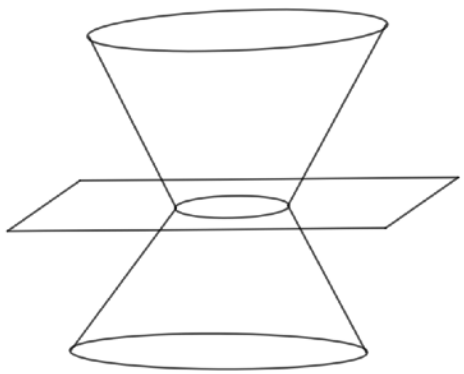
Quantum 5 p 520 (2021)

1st Sept 2021, ETH Zürich
Time in Quantum Theory Workshop

No signalling from the future: An OPT is **causal** if the probabilities of an operation do not depend on the choice of any *later* operation.





Relativistic Causality: A change in the initial data in a region *S*, does not produce any change in the regions outside the causal *past and future* of *S*.

Starting tension

Does quantum uncertainty imply time orientation?

No.

Then why are certain formulations of quantum theory time-oriented?

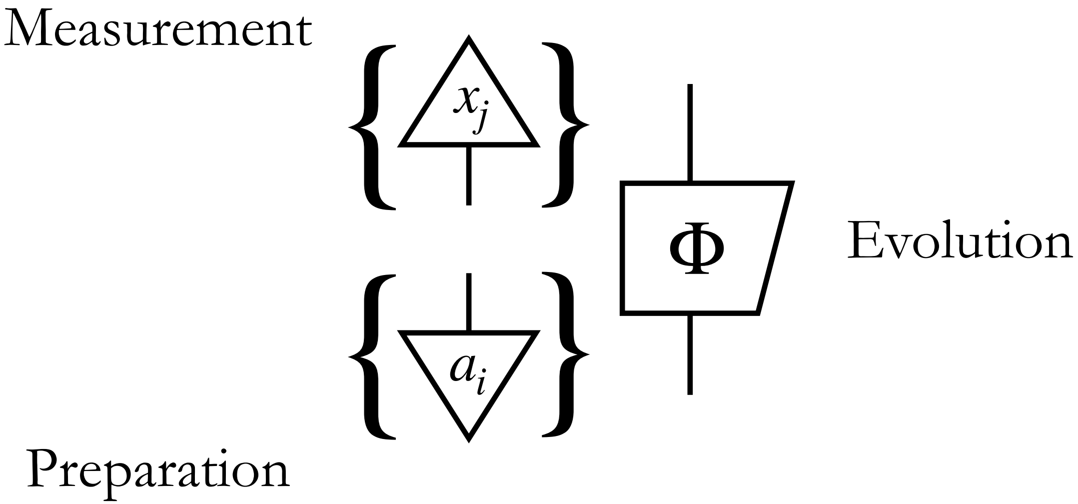
Reconstructions

Lucien Hardy, "Quantum Theory From Five Reasonable Axioms," (2001), arXiv:quant-ph/0101012.
Borivoje Dakic and Časlav Brukner, "Quantum theory and beyond: Is entanglement special?" (2009), arXiv:0911.0695 [quant-ph].
Lluís Masanes and Markus P. Müller, "A derivation of quantum theory from physical requirements," New Journal of Physics **13**, 063001 (2011).
G. Chiribella, G. M. D'Ariano, and P. Perinotti, "Informational derivation of Quantum Theory," Physical Review A **84**, 012311 (2011), arXiv:1011.6451.
Lucien Hardy, "Reconstructing quantum theory," (2013), arXiv:1303.1538 [gr-qc, physics:hep-th, physics:quant-ph].
Philipp A. Höhn, "Toolbox for reconstructing quantum theory from rules on information acquisition," Quantum **1**, 38 (2017), arXiv:1412.8323.
Philipp A. Höhn and Christopher Wever, "Quantum theory from questions," Physical Review A **95**, 012102 (2017), arXiv:1511.01130.
John H. Selby, Carlo Maria Scandolo, and Bob Coecke, "Reconstructing quantum theory from diagrammatic postulates," arXiv:1802.00367 [quant-ph] (2018), arXiv:1802.00367 [quant-ph].
Ding Jia, "Quantum from principles without assuming definite causal structure," Physical Review A **98**, 032112 (2018), arXiv:1808.00898.
Robert Oeckl, "A local and operational framework for the foundations of physics," Advances in Theoretical and Mathematical Physics **23**, 437–592 (2019), arXiv:1610.09052.

Plan

- Prediction and Postdiction
  - Closed quantum systems
  - Open Quantum Systems
  - Time-Reversal Symmetry
- Quantum operations
  - Review
  - Prediction and Postdiction
  - Arrow of inference, not the arrow of time
- Final remarks

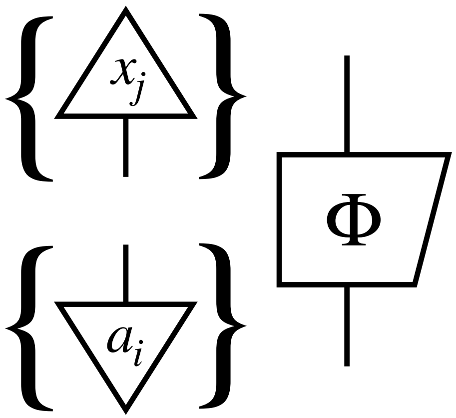
Two Games



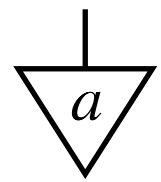
Prediction vs Postdiction

Prediction: Given a preparation, a test and the result of the preparation, calculate the probabilities of the outcomes of the test.

Given



and

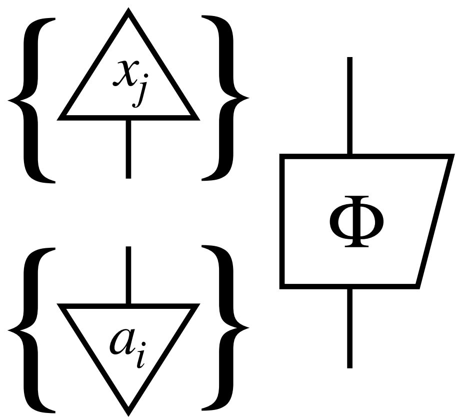


find $P_{pre}(x_j \mid a, \Phi)$

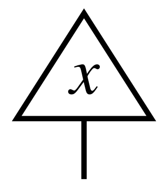
Prediction vs Postdiction

Postdiction: Given a preparation, a test and the result of the *test*, calculate the probabilities of the outcomes of the *preparation*.

Given



and



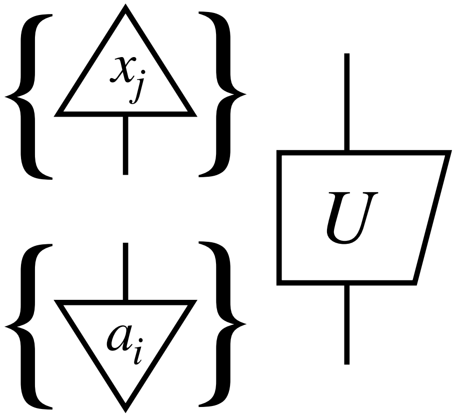
find $$P_{post}(a_i \,|\, x, \Phi)$$

Symmetry of Physical Laws. Part III. Prediction and Retrodiction
Satosi Watanabe
Rev. Mod. Phys. **27**, 179 – Published 1 April 1955

Quantum retrodiction in open systems
David T. Pegg, Stephen M. Barnett, and John Jeffers
Phys. Rev. A **66**, 022106 – Published 12 August 2002

Closed Systems

Given



Born rule

$$P_{pre}(x\,|\,a, U) = |\langle x\,|\,U\,|\,a\rangle|^2$$

Bayes' theorem

$$P_{post}(a\,|\,x, U) = \frac{P_{pre}(x\,|\,a, U)P(a)}{P(x)}$$

What are $P(a)$ and $P(x)$?

Closed Systems

$P(a)$ and $P(x)$ are *a priori* probabilities.

We only know $\{a_i\}$

Prior $P(a) = \dfrac{1}{d}$

Data $P(x) = \sum_{i=1}^{d} P_{pre}(x \mid a_i, U) P(a_i) = \sum_{i=1}^{d} \left| \langle x \mid U \mid a_i \rangle \right|^2 \cdot \dfrac{1}{d} = \dfrac{1}{d}$

$$P_{post}(a \mid x, U) = \frac{P_{pre}(x \mid a, U) P(a)}{P(x)} = P_{pre}(x \mid a, U)$$

Closed Systems

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$$P_{post}(a \mid x, U) = \left| \langle x \mid U \mid a \rangle \right|^2 = P_{pre}(x \mid a, U)$$

Time agnostic probabilities

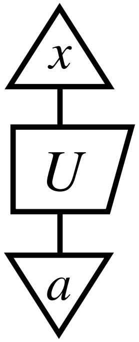
A process **Φ** is **inference symmetric** if:

$$P_{pre}(x_j \,|\, a_i, \Phi) = P_{post}(a_i \,|\, x_j, \Phi)$$

for any choice of bases.

Unitary evolution is inference symmetric $\Longrightarrow$ time agnostic.

$$P_{pre}(x \,|\, a, U) =$$

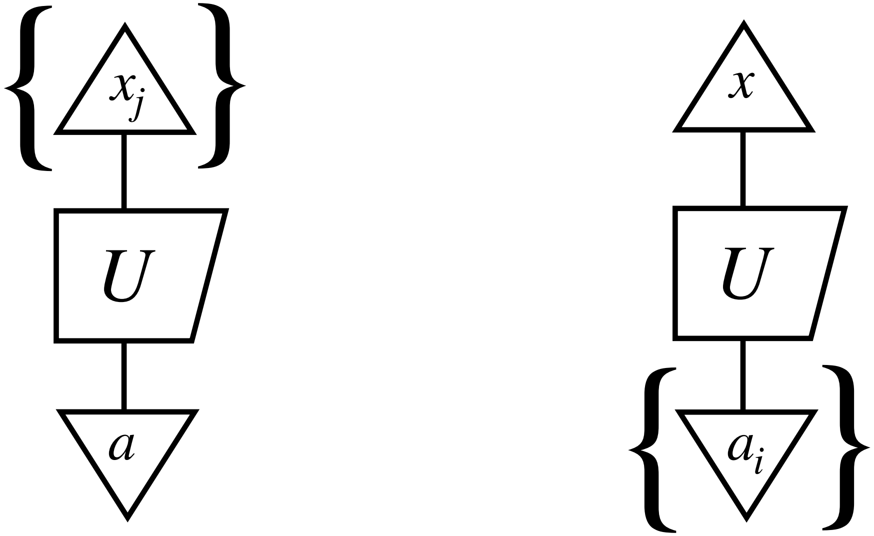


$$= P_{post}(a \,|\, x, U)$$

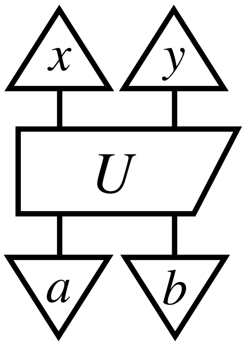
Time agnostic probabilities

Uniform prior is necessary for the above result.

But this is natural!



Open Systems

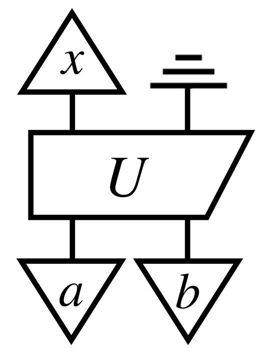


$$A \otimes B \equiv X \otimes Y$$

$$P_{pre}(xy\,|\,ab, U) = P_{post}(ab\,|\,xy, U)$$

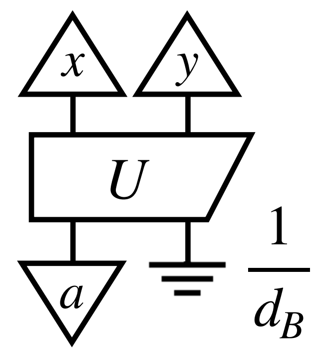
$$P_{pre}(x\,|\,ab, U) = \sum_{i=1}^{d_Y} P_{pre}(xy_i\,|\,ab, U)$$

$$= \mathrm{tr}\,\left((|x\rangle\langle x| \otimes I_Y)\; U[\,|ab\rangle\langle ab|\,]\right)$$

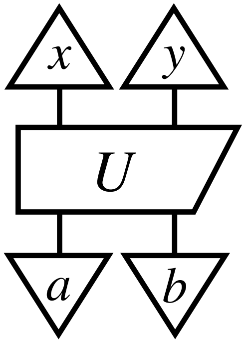


$$P_{pre}(xy\,|\,a, U) = \frac{1}{d_B}\sum_{i=1}^{d_B} P_{pre}(xy\,|\,ab_i, U)$$

$$= \mathrm{tr}\,\left(\,|xy\rangle\langle xy|\; U\left[\,|a\rangle\langle a| \otimes \frac{1}{d_B} I_B\right]\right)$$



Open Systems



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$$P_{pre}(xy\,|\,ab, U) = P_{post}(ab\,|\,xy, U)$$

$$P_{post}(ab\,|\,x, U) = \frac{1}{d_Y}\sum_{i=1}^{d_Y} P_{post}(ab\,|\,xy_i, U) = \frac{1}{d_Y}\sum_{i=1}^{d_Y} P_{pre}(xy_i\,|\,ab, U) = \frac{1}{d_Y} P_{pre}(x\,|\,ab, U)$$

$$P_{post}(a\,|\,xy, U) = \sum_{i=1}^{d_B} P_{post}(ab_i\,|\,xy, U) \quad = \sum_{i=1}^{d_B} P_{pre}(xy\,|\,ab_i, U) = d_B P_{pre}(xy\,|\,a, U)$$

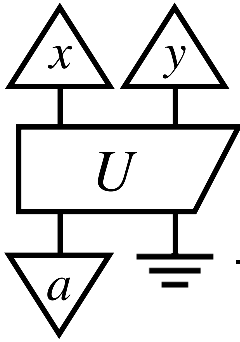
Direction of inference

$$P_{pre}(x \mid ab, U) = d_Y P_{post}(ab \mid x, U)$$

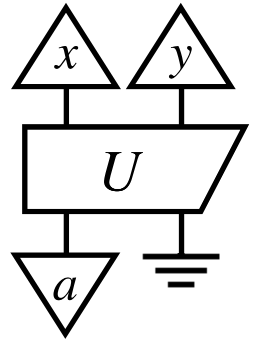
$$P_{pre}(xy \mid a, U) = \frac{1}{d_B} P_{post}(a \mid xy, U)$$

Prediction and postdiction simply related.

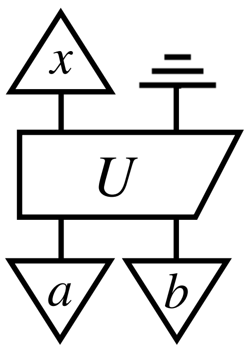
Direction of inference

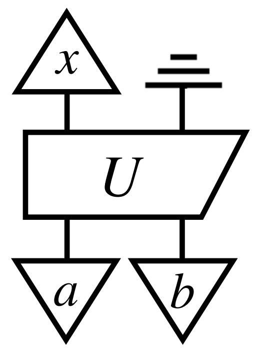
$P_{pre}(xy\,|\,a, U) =$ 

$\frac{1}{d_B}$



$= P_{post}(a\,|\,xy, U)$

$P_{pre}(x\,|\,ab, U) =$ 



$\frac{1}{d_Y} = P_{post}(ab\,|\,x, U)$

The direction of inference determines the normalisation of the identity.

Time-Reversal

Passive: Describe physical events in reversed order.

$\Longrightarrow$ swaps prediction and postdiction

Active: Find a process that undoes the original process.

$\Longrightarrow$ map to a new pair of games

$P_{pre}(x \mid a, U)$ ⟷ **Passive** ⟷ $P_{post}(a \mid x, U)$

Active ↕ $|\langle x | U | a \rangle|^2$ ↕ **Active**

$P_{pre}(a \mid x, U^{\dagger})$ ⟷ **Passive** ⟷ $P_{post}(x \mid a, U^{\dagger})$

Time-Reversal

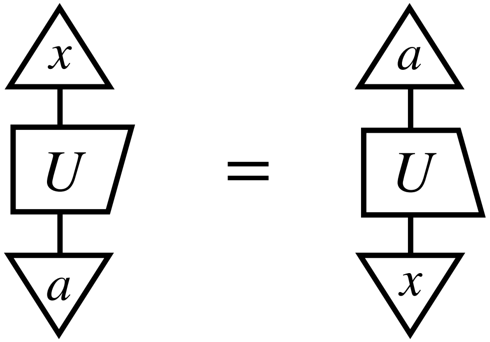
Passive: Describe physical events in reversed order.

$\Longrightarrow$ swaps prediction and postdiction

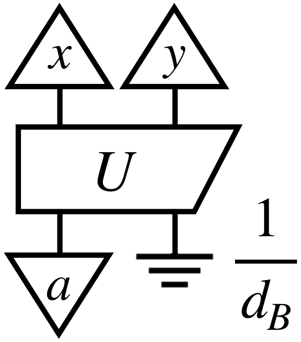
Active: Find a process that undoes the original process.

$\Longrightarrow$ map to a new pair of games

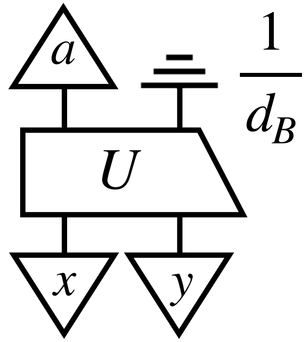
$$P_{pre}(a\,|\,x, U^\dagger) = |\langle a\,|\,U^\dagger\,|\,x\rangle|^2 = |\langle x\,|\,U\,|\,a\rangle|^2 = P_{pre}(x\,|\,a, U)$$



$$P_{pre}(xy\,|\,a, U) =$$

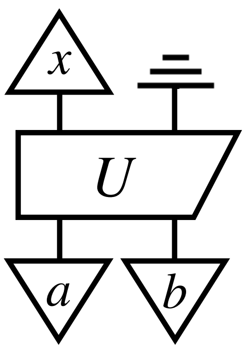


$$=$$

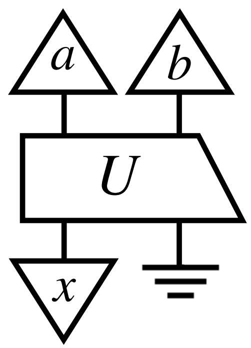


$$= P_{post}(xy\,|\,a, U^{\dagger})$$

$$P_{pre}(x\,|\,ab, U) =$$



$$=$$



$$= P_{post}(x\,|\,ab, U^{\dagger})$$

Plan

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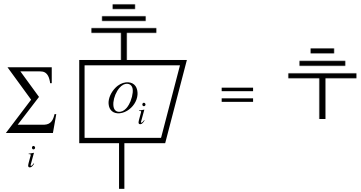
  - Arrow of Inference, not the arrow of time
- Conclusion

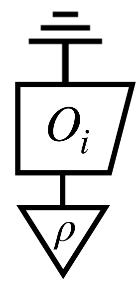
An **operation** $\mathcal{O}^{X\to A}$ is a set $\{O_i\}$ of completely positive trace non-increasing maps from linear operators on X to linear operators on A. Satisfying:

$$\mathrm{tr} \sum_i O_i[\rho] = \mathrm{tr}\ \rho$$



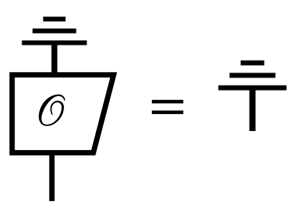
When the operation $\mathcal{O}^{X\to A}$ is applied to a system in state ρ, the event labelled by i happens with probability:

$$P\left(i\,|\,\rho, \mathcal{O}^{X\to A}\right) = \mathrm{tr}\ O_i[\rho] =$$



resulting in the new state $$\rho_i := \frac{O_i[\rho]}{\mathrm{tr}\ O_i[\rho]}$$

If we don't know the outcome $$\mathcal{O}[\rho] = \sum_i O_i[\rho]$$



Operations

Two operations $\mathscr{O}^{X\to A}$ and $\mathscr{M}^{A\to B}$ can be composed in **sequentially** with

$$\mathscr{M} \circ \mathscr{O} = \{M_j \circ O_i\}$$

with the probability of the event ij given by

$$M_j[O_i[\rho]] = M_j\left[\frac{O_i[\rho]}{\mathrm{tr}\, O_i[\rho]}\right] \mathrm{tr}\, O_i[\rho]$$

$$P\left(ij\,|\,\rho, \mathscr{M} \circ \mathscr{O}\right) := \mathrm{tr}\, M_j[O_i[\rho]] = P(j\,|\,\rho_i, \mathscr{M})P(i\,|\,\rho, \mathscr{O})$$

Two operations $\mathscr{O}^{X\to A}$ and $\mathscr{M}^{A\to B}$ can also be composed **in parallel** using the tensor product structure of Hilbert spaces.

$\Longrightarrow$ Quantum operations form a symmetric monoidal category. Many interesting results follow. Can generalise...

Operations

An operation $\mathscr{P}^{\mathbb{C}\to A}$ is called a **preparation**. It can be represented by a set of $\{\rho_i\}$ positive-semidefinite hermitian operators on A such that $\sum_i \operatorname{tr} \rho_i = 1$.

An operation $\mathscr{T}^{A\to\mathbb{C}}$ is called an **effect.** It can be represented by a set $\{\sigma_i\}$ of positive-semidefinite operators on A such that $\sum_i \sigma_i = I$ (a POVM).

An operation with a single outcome is deemed **deterministic**.

A deterministic preparation is called a **state**.

A deterministic operation is called a **channel**.

There is only one deterministic effect: taking the trace, aka the discard

Operations are time-oriented

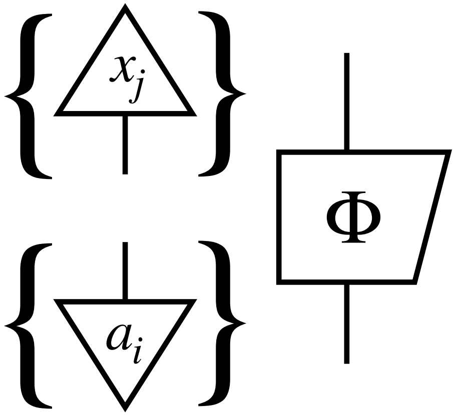
The state of a system always depends on *past* operations.

One can choose the state of the system *before* an operation, but *not after*.

All probabilities are *prediction* probabilities.

$$\rho \longmapsto \rho_i = \frac{O_i[\rho]}{\mathrm{tr}\ O_i[\rho]} \qquad\qquad P(i\,|\,\rho, \mathcal{O}) = \mathrm{tr}\ O_i[\rho]$$

Given



Generalised Born rule

$$P_{pre}(x\,|\,a,\Phi) = \mathrm{tr}\,|x\rangle\langle x|\,\Phi[\,|a\rangle\langle a|\,]$$

Bayes' theorem

$$P_{post}(a\,|\,x,\Phi) = \frac{P_{pre}(x\,|\,a,\Phi)P(a)}{P(x)}$$

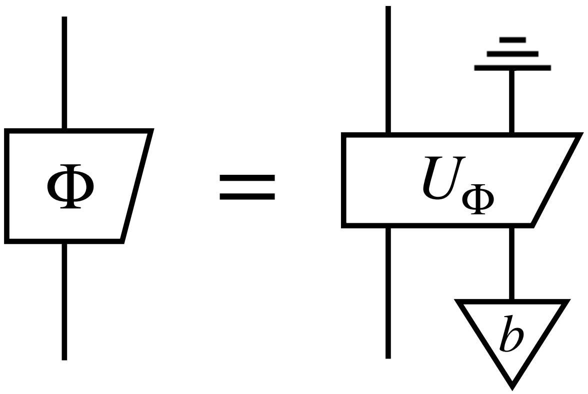
Prior $P(a) = \frac{1}{d_A}$

Data $P(x) = \sum_{i=1}^{d_A} \frac{1}{d_A} P_{pre}(x \mid a_i, \Phi) = \frac{1}{d_A} \mathrm{tr}\, |x\rangle\langle x| \, \Phi[\mathbb{I}_A]$

$$P_{post}(a \mid x, \Phi) = \frac{\mathrm{tr}\, |x\rangle\langle x| \, \Phi[\, |a\rangle\langle a| \,]}{\mathrm{tr}\, |x\rangle\langle x| \, \Phi[\mathbb{I}_A]} = \frac{P_{pre}(x \mid a, \Phi)}{\mathrm{tr}\, |x\rangle\langle x| \, \Phi[\mathbb{I}_A]}$$

Channels are not inference-symmetric in general.

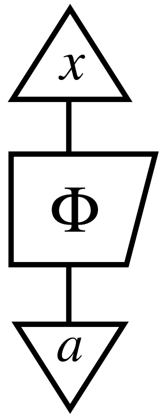
Stinespring Dilation: Any quantum channel can be understood in terms of a unitary interaction with an ancilla system.

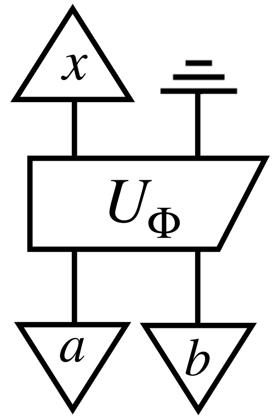


$$\Phi[\rho] = \mathrm{tr}_Y\, U_\Phi[\rho \otimes |b\rangle\langle b|]$$

This allows us to understand the inference asymmetry of the quantum channels.

Purification

$P_{pre}(x \,|\, a, \Phi) =$ 

$=$ 

$= P_{pre}(x \,|\, ab, U_{\Phi})$

$$P_{post}(a \,|\, x, \Phi) = P_{post}(a \,|\, xb, U_{\Phi})$$

Purification

$$P_{post}(a\,|\,xb, U_\Phi) = \frac{P_{post}(ab\,|\,x, U_\Phi)}{P_{post}(b\,|\,x, U_\Phi)}$$

$$P(a|b) = \frac{P(ab)}{P(b)}$$

$$P_{post}(ab\,|\,x, U_\Phi) = \frac{1}{d_Y} P_{pre}(x\,|\,ab, U_\Phi)$$

$$P_{post}(b\,|\,x, U_\Phi) = \frac{d_A}{d_Y} P_{pre}(x\,|\,b, U_\Phi)$$

$$P_{post}(a\,|\,xb, U_\Phi) = \frac{P_{pre}(x\,|\,ab, U_\Phi)}{d_A P_{pre}(x\,|\,b, U_\Phi)}$$

Purification

$$P_{post}(a\,|\,xb, U_\Phi) = \frac{P_{pre}(x\,|\,ab, U_\Phi)}{d_A P_{pre}(x\,|\,b, U_\Phi)}$$

$$P_{pre}(x\,|\,ab, U_\Phi) = P_{pre}(x\,|\,a, \Phi)$$

$$P_{pre}(x\,|\,b, U_\Phi) = \sum_{i=1}^{d_A} \frac{1}{d_A} P_{pre}(x\,|\,a_i b, U_\Phi) = \frac{1}{d_A} \sum_{i=1}^{d_A} P_{pre}(x\,|\,a_i, \Phi)$$

$$P_{post}(a\,|\,xb, U_\Phi) = \frac{P_{pre}(x\,|\,ab, U_\Phi)}{\mathrm{tr}\,|x\rangle\langle x|\,\Phi[\mathbb{1}_A]} = P_{post}(a\,|\,x, \Phi)$$

Purification

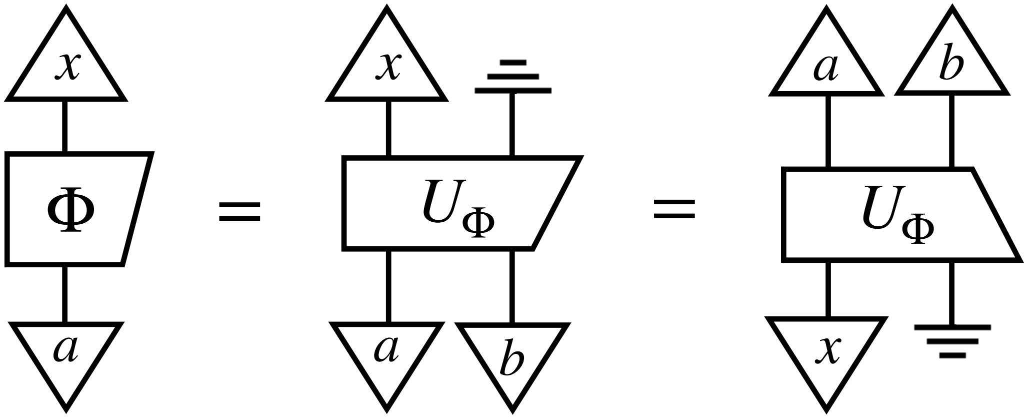
$$P_{post}(a|x,\Phi) = P_{post}(a|xb,U_{\Phi})$$

The inference asymmetry of quantum channels is understood as an asymmetry in the inference data.

The specification of the channel Φ implicitly contains information about the state of an ancilla system B, which is assumed known.

*Similar technique can be applied to mixed-state preparations and POVMs. See paper.

Quantum channels towards the past



$$= P_{post}(x \,|\, ab, U_\Phi^\dagger)$$

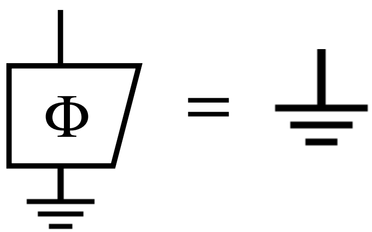
$\Longrightarrow$ quantum channels can represent postdiction probabilities

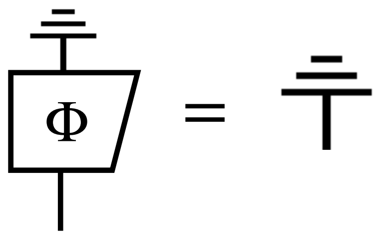
Inference Symmetric Channels

$$P_{post}(a\,|\,x,\Phi) = \frac{P_{pre}(x\,|\,a,\Phi)}{\mathrm{tr}\,|x\rangle\langle x|\,\Phi[\mathbb{1}_A]}$$

$$P_{post}(a\,|\,x,\Phi) = P_{pre}(x\,|\,a,\Phi)$$

$$\Longleftrightarrow \qquad \Phi[\mathbb{1}_A] = \mathbb{1}_X$$





Φ is Inference-Symmetric $\Longleftrightarrow$ Φ admits a time reversal

Symmetries of quantum evolutions

Giulio Chiribella, Erik Aurell, and Karol Życzkowski
Phys. Rev. Research **3**, 033028 – Published 6 July 2021

Two Maxims

There exists a unique deterministic effect.

The choice of an operation does not affect the probabilities of the outcome of an earlier operation.

Maxim 1

There exists a unique deterministic effect.

Mathematically correct: the trace is the only CPTP map to the trivial space.

Physically correct: there is fundamental unpredictability in QM.

But not a difference between past and future: there is fundamental un*post*dictability in QM.

$$P_{post}(a\,|\,x,\Phi) = \frac{\mathrm{tr}\,|x\rangle\langle x|\,\Phi[\,|a\rangle\langle a|\,]}{\mathrm{tr}\,|x\rangle\langle x|\,\Phi[\mathbb{1}_A]}$$

Maxim 2

The choice of an operation does not affect the probabilities of the outcome of an earlier operation.

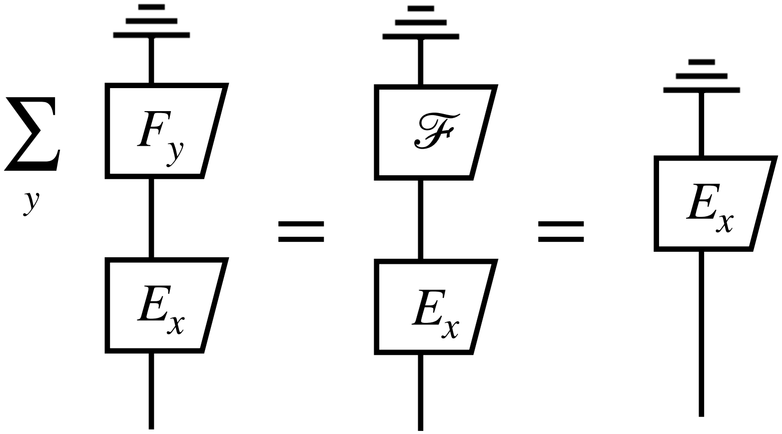
Mathematically correct: a consequence of conservation of probabilities.

Physically correct: experimentally corroborated.

But not a difference between past and future: difference between known and unknown

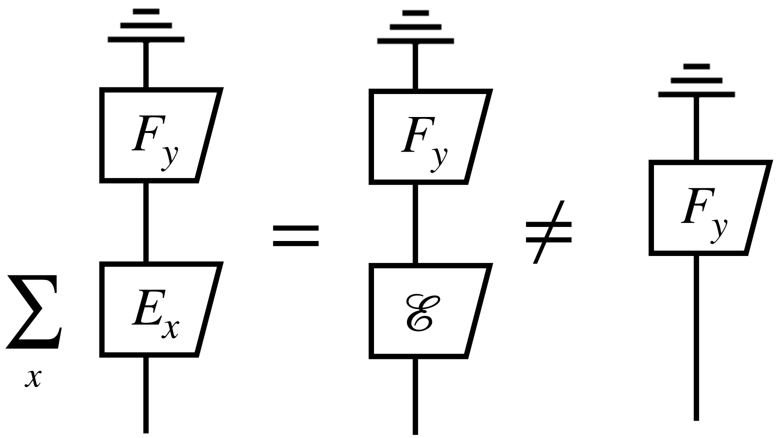
Maxim 2

$$P(x\,|\,\rho,\mathscr{F}\circ\mathscr{E}) = \sum_y \mathrm{tr}\, F_y\big[E_x[\rho]\big] = \mathrm{tr}\, E_x[\rho] = P(x\,|\,\rho,\mathscr{E})$$

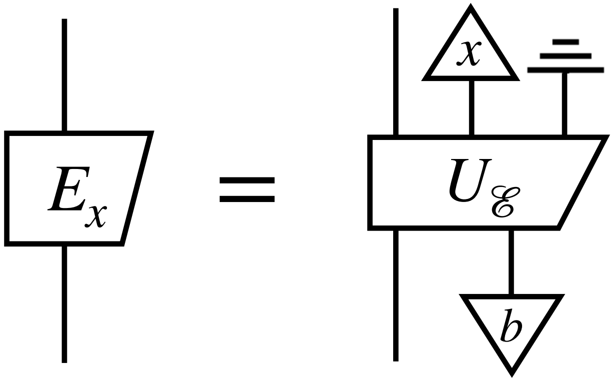


Maxim 2

$$P(y\,|\,\rho, \mathscr{F}\circ\mathscr{E}) = \sum_x \operatorname{tr} F_y\big[E_x[\rho]\big] = \operatorname{tr} F_y\big[\mathscr{E}[\rho]\big] \neq \operatorname{tr} F_y[\rho]$$

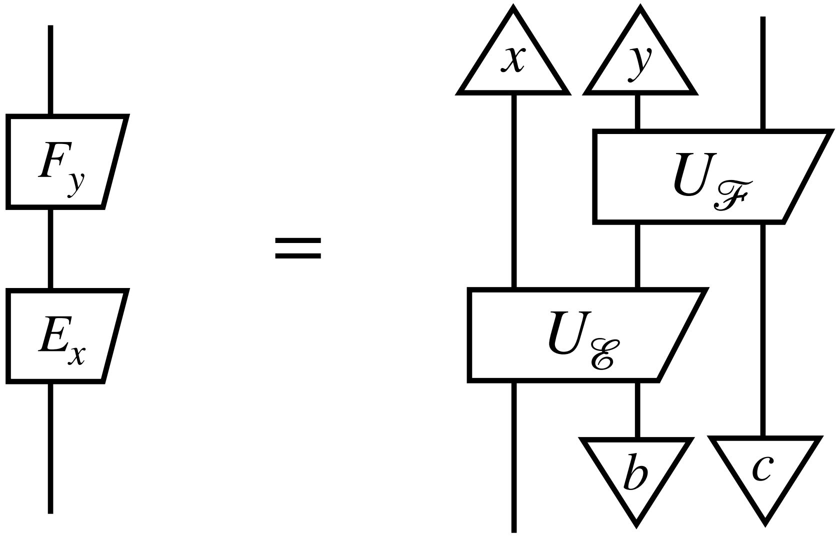


Ozawa dilation: Any quantum operation can be understood in terms of a unitary interaction with an ancilla system, and a projective measurement

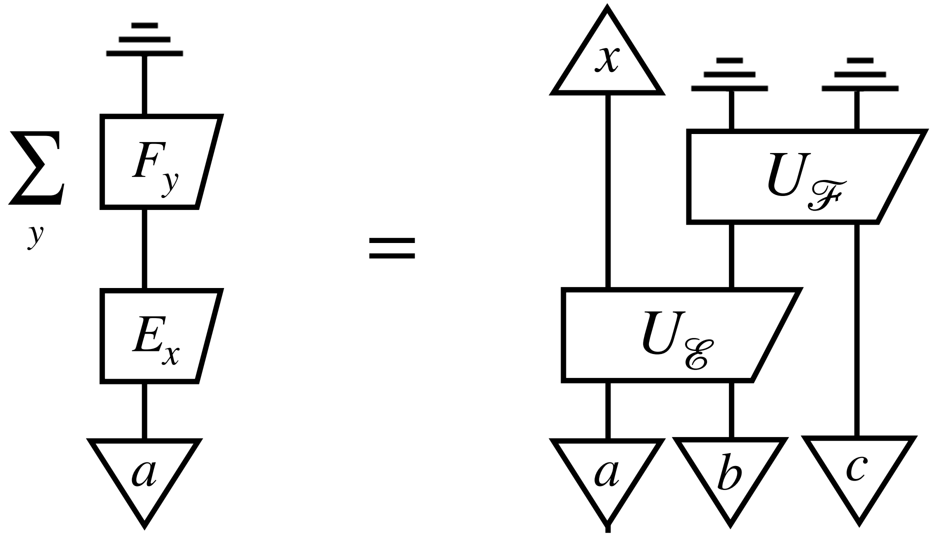


$$E_x[\rho] = \mathrm{tr}_Y\Big[\big(I \otimes |x\rangle\langle x| \otimes I_Y\big)\, U_\Phi[\rho \otimes |b\rangle\langle b|\,]\Big]$$

Maxim 2

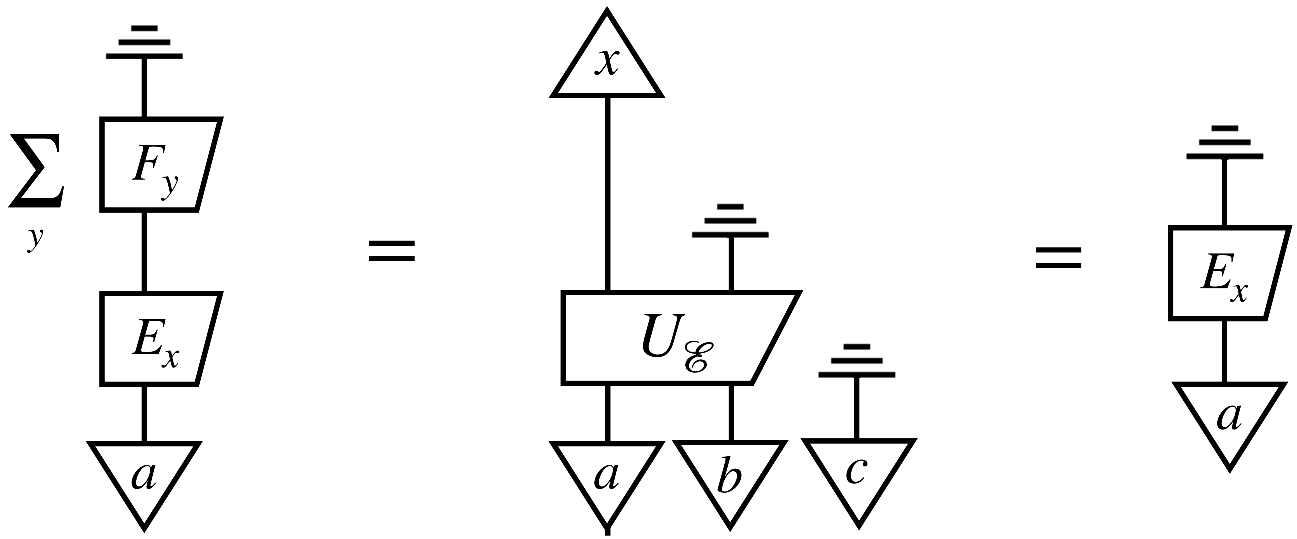


Maxim 2



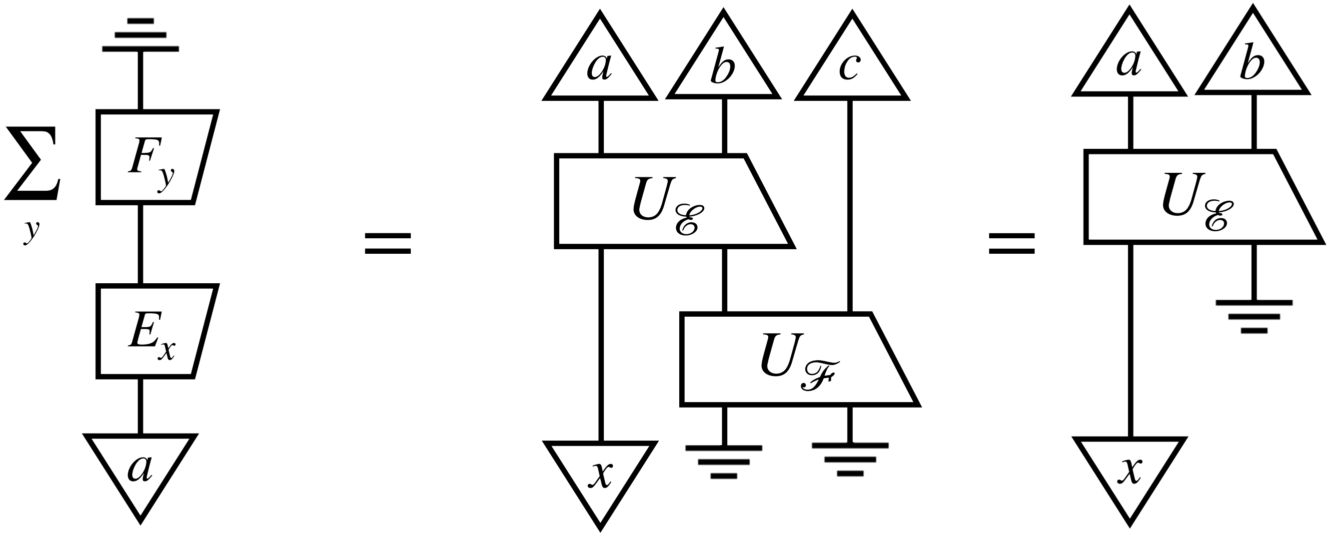
$$P(x\,|\,a, \mathscr{F} \circ \mathscr{E}) = P_{pre}(x\,|\,abc, U_{\mathscr{F}} \circ U_{\mathscr{E}})$$

Maxim 2



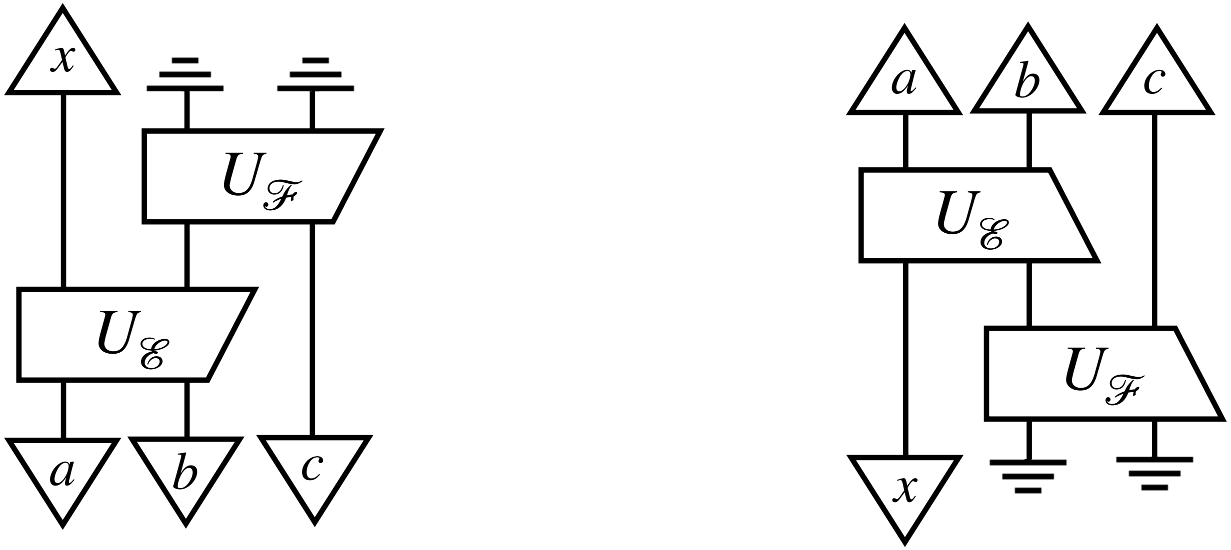
$$P(x\,|\,a, \mathscr{F}\circ\mathscr{E}) = P_{pre}(x\,|\,abc, U_{\mathscr{F}}\circ U_{\mathscr{E}}) = P(x\,|\,a, \mathscr{E})$$

Maxim 2



$$P(x\,|\,a, \mathscr{F} \circ \mathscr{E}) = P_{post}(x\,|\,abc, U^{\dagger}_{\mathscr{E}} \circ U^{\dagger}_{\mathscr{F}}) = P_{post}(x\,|\,ab, U^{\dagger}_{\mathscr{E}})$$

Maxim 2



No signalling from the further unknown.

Plan

- Prediction and Postdiction
  - Closed quantum systems
  - Open Quantum Systems
  - Time-Reversal Symmetry
- Quantum operations
  - Review
  - Prediction and Postdiction
  - Arrow of Inference, not the arrow of time
- Conclusion

Why the asymmetry?

There are two asymmetric aspects:

- We are interested in prediction
- We consider time-asymmetric inference problems

Both may be understood in terms of thermodynamics:

- We remember the past, and not the future
- We make choices that affect the future, not the past

Price, *Time's arrow & Archimedes' point*, Oxford University Press (1997)
Mlodinow and Brun, *Relation between the psychological and thermodynamic arrows of time*. Phys. Rev. E **89**, (2014)
Rovelli, *Agency in Physics*. arXiv:2007.05300 (2020)
Rovelli, *Memory and entropy*. arXiv:2003.06687 (2020)
Ismael, *How physics makes us free*, Oxford University Press (2016)

Why the asymmetry?

Time-asymmetry due to the users of QM.

QI is about correlations established between agents.

The agent is not explicitly modelled by the theory, but *represented* in the mathematical objects in the theory.

Towards a time symmetric reconstruction

[Submitted on 31 Mar 2021]

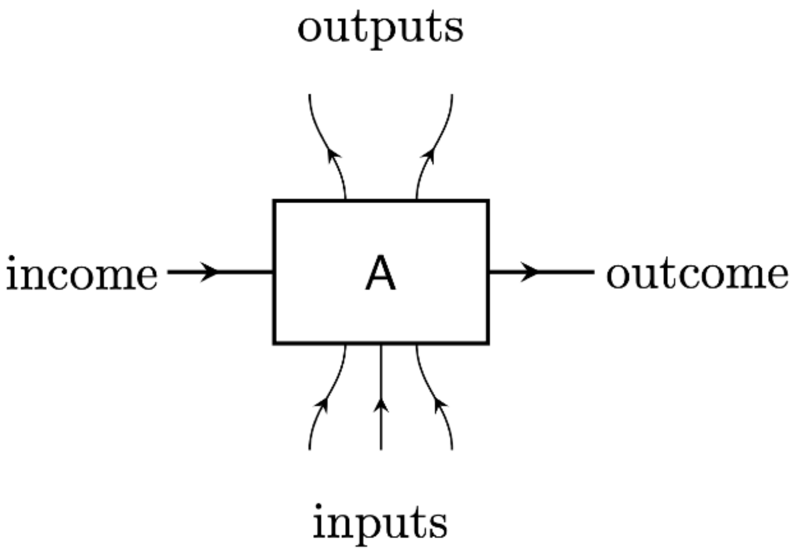
Time Symmetry in Operational Theories

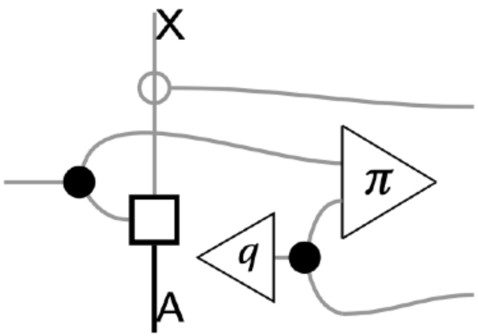
Lucien Hardy arXiv:2104.00071

[Submitted on 7 Sep 2020 (v1), last revised 19 May 2021 (this version, v3)]

Unscrambling the omelette of causation and inference: The framework of causal-inferential theories arXiv:2009.03297

David Schmid, John H. Selby, Robert W. Spekkens





Today at 15:15!

Thank you

To be continued....

Thank you for listening!