3 Limits, Part II (Edge Cases)

3.1 Absence of limits

What does it mean to say L is not a limit of $f(x)$ at a? It flows out of the definition– there exist some ϵ such that for any δ there exists an x in $0 < |x-a| < \delta$ such that $|f(x) - L| \geq \epsilon$.

A stronger version is to say there is no limit of $f(x)$ at a. To do that we must prove that *any* L is not a limit of $f(x)$ at a.

Example: Absolute value fraction

Consider $f(x) = \frac{x}{|x|}$. It's easy to see that

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

We will show there is no limit of $f(x)$ near 0.

Weak version. First, let's prove a weak version– that $\lim_{x\to 0} f(x) \neq 0$. That is easy. Pick some reasonably small epsilon, say $\epsilon = \frac{1}{10}$. We must show that for any δ there exists an x in $0 < |x-a| < \delta$ such that $|f(x) - 0| \geq \frac{1}{10}$.

Let's pick some arbitrary x out of our permitted interval, say $x = \delta/2$. Then

$$|f(x) - 0| = |f(\delta/2)| = \left|\frac{\delta/2}{|\delta/2|}\right| = 1 \geq \frac{1}{10}$$

Strong version. Now we prove that $\lim_{x\to 0} f(x) \neq L$ for *any* L. Sticking with $\epsilon = \frac{1}{10}$ we proceed as follows.

If $L < 0$ take $x = \delta/2$. Then

$$|f(x) - L| = |f(\delta/2) - L| = \left|\frac{\delta/2}{|\delta/2|} - L\right| = |1 - L| > \frac{1}{10}$$

Similarly if $L \geq 0$ take $x = -\delta/2$. Then

$$|f(x) - L| = |f(-\delta/2) - L| = \left|\frac{-\delta/2}{|-\delta/2|} - L\right| = |-1 - L| > \frac{1}{10}$$

Example: Dirichlet function

The *dirichlet* function f is defined as follows:

$$f(x) = \begin{cases} 1 & \text{for rational } x, \\ 0 & \text{for irrational } x. \end{cases}$$

We prove $\lim_{x\to a} f(x)$ does not exist for any a.

Proof. Let $\epsilon = \frac{1}{10}$. Suppose for contradiction there exists L such that $\lim_{x\to a} f(x) = L$. There are two possibilities: either $L \leq \frac{1}{2}$ or $L > \frac{1}{2}$.

First suppose $L \leq \frac{1}{2}$. Pick any rational x from the interval $0 < |x - a| < \delta$. Then $|f(x) - L| = |1 - L| \geq \frac{1}{2}$. Thus $|f(x) - L| \geq \frac{1}{10}$.

Similarly, suppose $L > \frac{1}{2}$. Pick any irrational x from the interval $0 < |x-a| < \delta$. Then $|f(x) - L| = |0 - L| > \frac{1}{2}$. Thus $|f(x) - L| \geq \frac{1}{10}$.

Thus $\lim_{x\to a} f(x)$ does not exist for any a, as desired.

3.2 One-sided limits

We have seen that the following function has no limit approaching 0:

$$f(x) = \begin{cases} -1 & x < 0, \\ 1 & x > 0 \end{cases}$$

However, f has properties around 0 we may want to be able to formally describe. First, intuitively f approaches -1 as we approach zero from the left (from "below"). Not surprisingly, a notation for this exists:

$$\lim_{x\to 0^-} f(x) = -1$$

If we take $l = -1$, this notation compiles down to the following definition. For every $\epsilon > 0$ there exists $\delta > 0$ such that $0 < a - x < \delta$ implies $|f(x) - l| < \epsilon$ for all x. This is our usual limit definition, except instead of looking at both sides of a, the inequality $0 < a - x$ says $x < a$ (i.e. we look from left of a).

Second, intuitively f approaches 1 as we approach zero from the right (from "above"). The notation for this is:

$$\lim_{x\to 0^+} f(x) = 1$$

If we take $l = 1$, the definition is as follows. For every $\epsilon > 0$ there exists $\delta > 0$ such that $0 < x - a < \delta$ implies $|f(x) - l| < \epsilon$ for all x. Again, this is our usual limit definition, except instead of looking at both sides of a, the inequality $0 < x - a$ says $x > a$ (i.e. we look from right of a).

3.3 Limits at infinity

Consider the function $f(x) = \frac{1}{x}$. Clearly as x gets very large, $f(x)$ trends toward zero. Again, we have a notation that encodes this property of f:

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

Take $l = 0$, and this compiles down to the following definition. For every $\epsilon > 0$ there is a number N such that $|f(x) - l| < \epsilon$ for all $x > N$.

Intuitively, for any ϵ, $f(x)$ will get within ϵ of the limit for x large enough. Here we simply produce a large enough N instead of δ.

3.4 Infinite limits

Consider the function $f(x) = \frac{1}{x^2}$. Near zero f shoots up, and again, we want to be able to encode that. The notation for this property is

$$\lim_{x \to 0} f(x) = \infty$$

This compiles down to the following definition. Given any $M > 0$ there exists $\delta > 0$ such that $0 < |x - a| < \delta$ implies $f(x) > M$ for all x. Intuitively, given an arbitrarily large $f(x) = M$ we can produce a bound on the x-axis, within which $f(x)$ is never smaller than M.

Example. Suppose we want to prove $\lim_{x \to 0} \frac{1}{x^2} = \infty$. Let $M > 0$ be given. We must produce $\delta > 0$ such that $0 < |x| < \delta$ implies $\frac{1}{x^2} > M$ for all x. Suppose we fix $|x| < \frac{1}{\sqrt{M}}$. Then:

$$\begin{aligned} |x| &< \frac{1}{\sqrt{M}} && \text{note } M > 0 \\ \implies x^2 &< \frac{1}{M} \\ \implies \frac{1}{x^2} &> M \end{aligned}$$

Thus $\delta \leq \frac{1}{\sqrt{M}}$ implies $\frac{1}{x^2} > M$ as desired.