MSc Project Report
Asymptotic Self-Similarity in the Mandelbrot Set

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Abstract

This work contains a brief overview of the theory of complex dynamics necessary to understand a result by Tan Lei, namely that the Mandelbrot set is asymptotically self-similar about certain points on its boundary. Tan Lei proves this by first proving a general statement about continuous maps on $\mathbb{C}^k$. Here the proof is abridged to avoid the general case and prove the result directly. This is done with the idea that the conceptual basis of the proof is made clearer.

Acknowledgements

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I also express gratitude to my family for having supported my studies and apologies for my absent-mindedness. I am also grateful to my dear friend Jack Jelfs for all the interesting conversations, not just about mathematics.

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Contents

Fractals capture and inspire the imagination of the general population, and thus play a special role in modern mathematics as some of the most recognised and appreciated mathematical objects. In this context, the Mandelbrot set plays a starring role, as any Google query for fractal images will show. Its boundary presents increasing beauty and complexity as revealed by the videos of continuous magnifications created by computer enthusiasts, sometimes reaching millions of views on video streaming websites. In these mind-bending visual journeys at various magnifications, the set often looks like itself at vastly different scales, while novel patterns, reminiscent of Azerbaijani textiles, seahorse tails and echinoderms, continually arise.

Figure 1: *Details of the Mandelbrot set juxtaposed with images from nature and textiles.* The images of the Mandelbrot set are still frames from a video by Michael Hogg [22], the other images are available online [23, 24, 25]

Self-similarity in fractals often is explicitly built into the instructions to generate the object. Notable examples are the Koch curve, and Sierpinski's gasket. Whatever deviations from exact self-similarity, such as random angles or deformations, are also included in the definition. A common definition of the Mandelbrot set, however, includes neither:

Definition 1. Let c be a complex number and $f_c : z \mapsto z^2 + c$. Then c is in the Mandelbrot set if the sequence $0, c, f_c(c), f_c(f_c(c)), \cdots$ is bounded.

This definition has the advantage of being simple to understand with a basic knowledge of complex numbers, and it easy to check that M is contained in the closed disk of radius 2 centred at the origin, providing a first way of approximating the Mandelbrot set with a simple computer program.

What is perhaps striking when one has seen a number of the magnifications, is to discover that, in the midst of all this complexity, there are indeed points around which the Mandelbrot set assumes a progressively self-similar character as a sequence of magnifications is performed. This happens at all points c such that the sequence $0, c, f_c(c), f_c(f_c(c)), \cdots$ initially wanders, then settles on a periodic cycle. These points are called Misiurewicz points, and are particular solutions of the polynomial equations:

$$f_c^p \circ f_c^l(0) = f_c^l(0)$$

It is remarkable that it is at all possible to find simplicity in the seeming chaos of the boundary of the Mandelbrot set. This project was undertaken to understand and explain this behaviour.

As with most questions about the Mandelbrot set, to work with the above definition is hopeless. To understand this and other features of the Mandelbrot set and its boundary, it is necessary look at it in the context of complex dynamics and its methods: a rich intersection of complex and functional analysis, topology and number theory.

The main result in this work was proved by Tan Lei in [3] (Theorem 5.1); this work hopes to elucidate the proof provided therein. A statement of the theorem is:

Theorem 1 (Tan Lei)**.** Let c_0 be a Misiurewicz parameter. Then the Mandelbrot set is asymptotically self-similar about c_0 and is asymptotically similar to the Julia set of the quadratic map f_{c_0} about c_0.

Section 1 provides a short historical context for complex analysis and the discovery of fractals within it. Section 2 will be dedicated to building the necessary background in complex analysis. In particular it will present the theory of normal families of functions of the complex plane, the results of which were fundamental in developing the theory of complex dynamics, which will be exposed in Section 3. In particular, the Fatou and Julia set of a holomorphic map will be introduced and many properties of these remarkable sets proven. This forms the background to the study of the Mandelbrot set, which will be properly defined and introduced in its natural context in Section 4, and some modern results reviewed. Section 5 contains the metric framework for the study of (asymptotic) self-similarity. Sections 6 to 8 follow closely [3], expanding and providing details in the proofs. Complex dynamics is a very rich field, and it was beyond the scope of this work to give a complete overview. Proofs that are instructive and reasonably short are included. The reader will be directed to references when suitable.

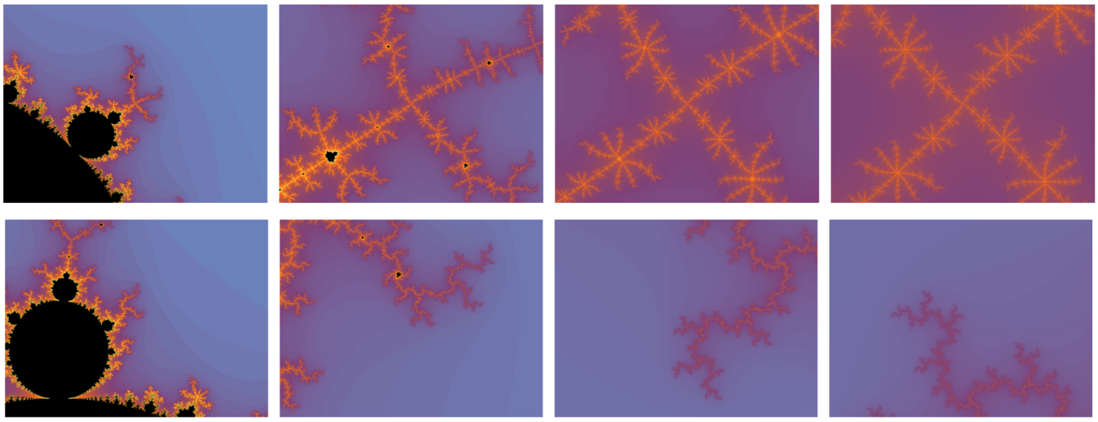


Figure 2: *Asymptotic self-similarity* Successive magnifications by factors of 10 about the points $0.366363 + 0.591534i$ and $0.0252242 + 0.805037i$, generated with Mathematica 11 by the author.

1 Historical Note

The subject of complex dynamics is the local and global behaviour of iterates $f, f^2, f^3 \cdots$ of a function f of the complex plane. A function f is thought as a *map* determining the position of a point at timestep $n + 1$ based on its position at timestep n. The sequence $\{z, f(z), f^2(z), \cdots\}$, where f^n denotes the n-fold composition of f with itself, is called *orbit* of z. Of immediate interest are then the solutions to the equation

$$z = f(z)$$

which are called the *fixed points* of f. The following theorem was already known by Schröder in the late nineteenth century:

Theorem 1.1. If a complex function f is holomorphic at a fixed point z and $|f'(z)| < 1$ then the orbit of all points in a neighbourhood of z will converge to z. This kind of fixed point is called an *attracting fixed point*. Similarly, if a point z satisfies

$$z = f^n(z)$$

it means that after n iterates, the orbit goes through z again: it is a *periodic* orbit. One can then look at the map $g(z) = f^n(z)$ to see if this orbit is attracting. These are instances of *local behaviour*, which concerns itself of arbitrarily small neighbourhoods of fixed or periodic points. Theorems of local behaviour arose in the late nineteenth century and first decade of the twentieth century in the works of Schröder , Cayley, Koenigs, Grévy and Leau, all before the advent of Set theory. A very different question is that of *global behaviour* where the orbit of an arbitrary point in the complex plane is considered. The meaningful treatment of this question had to wait the beginning of the twentieth century, set theory, and the works of Paul Montel, Pierre Fatou and Gaston Julia [1]. These mathematicians proved many results about the awesomely complex fractals arising in complex dynamics, without ever seeing them with their eyes.

Interest in iterations of complex functions is connected the study of convergence of the Newton method for determining the roots of a polynomial of any degree. Here an initial guess z_0 for the root is chosen, and then one generates a series of approximations $z_1, z_2, \cdots$ of the root by iterating the formula:

$$z_{n+1} = N(z_n) = z_n - \frac{f(z_n)}{f'(z_n)}$$

The global behaviour of Newton's method for a quadratic polynomial with distinct roots was understood by Schröder : the orbit of a point z converges to the closest root. If the initial guess it is equidistant from both roots, the orbit stays equidistant, and the orbit never converges to a finite solution. Thus the plane is divided in two *basins of attraction* and a line. In the neighbourhood of points on the line, the dynamics is sensitively dependent on initial conditions: orbits of points arbitrarily close to the line will converge to either root, or not converge at all.

The behaviour of Newton's method for a polynomial of third degree is illustrative of the complexity that can arise in global behaviour. Consider the iteration for finding the third roots of unity, the solutions of $z^3 - 1 = 0$. The map is

$$z \mapsto z - \frac{z^3 - 1}{3z^2} \tag{1}$$

We know that the solutions are the three points $1, e^{\pm\frac{2\pi}{3}i}$, and indeed guesses close enough will be converging to these. A first guess for the boundaries would be that the plane is divided in three sections, with straight lines originating from 0 as boundaries. Numerical studies show that this is not the case. In fact, according to [2], John Hubbard was teaching a class in elementary calculus at Orsay when he stumbled upon the question of which root does a guess converge to. In contrast with Julia and Fatou, Hubbard had the possibility to experiment with computers. He wrote a program that calculated the trajectory of each point of the plane, and colour coded the coordinates of that point according to which root the point converged to. As he increased the resolution, he was reportedly bewildered. There was no neat boundary between two regions: between any two regions between any two colours, lied the third. With further computing power, one can appreciate the fine structure of this boundary, called the *Julia set*.

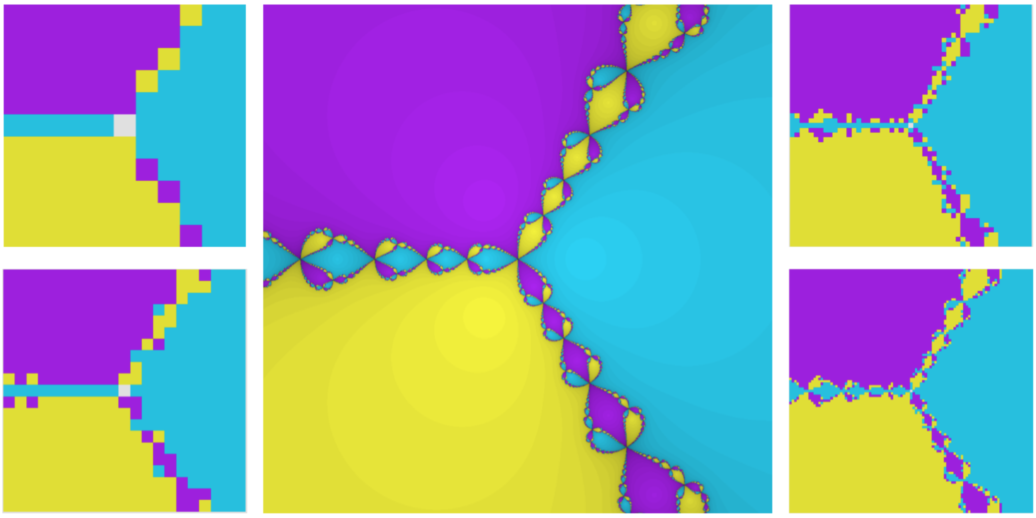


Figure 3: *A glimpse of Chaos* The basin of attractions for the dynamical system 1, with different resolutions

2 Complex Analysis

We present basic results of complex analysis, the terminology is standard and the reader is directed to references like [6] or [7] for details of proofs in Sections 2.1 and 2.2. In particular, the results about normal families are worked out in detail as they are fundamental in the study complex dynamics and it is possible to give a good overview here. For details about normal families of arbitrary Riemann surfaces, see Chapter 3 in [8].

In the following, $\mathbb{R}$ will denote the real numbers, $\mathbb{C}$ will denote the set of complex numbers, $\mathbb{D}$ the unit open disk centred at the origin and $\mathbb{D}_r(z)$ the open disk of radius $r > 0$ centred at point z.

2.1 Holomorphic functions

Definition 2. Given an open set U of $\mathbb{C}$, a function $f : U \to \mathbb{C}$ is *holomorphic* at $z_0 \in U$ if the first derivative

$$z \longmapsto f'(z) = \lim_{\epsilon \to 0} \frac{f(z+\epsilon) - f(z)}{\epsilon}$$

is well-defined and continuous for all z in a neighbourhood of z_0, independent on the specific way ϵ tends to 0. The function is *holomorphic on* U if it is holomorphic at all points in U.

Differentiability of a function of a complex variable is a much stronger requirement than differentiability of a function of a real variable. Indeed, we will see that if a function $f : U \to \mathbb{C}$ is holomorphic, then not only is it *smooth*, meaning that derivatives of any order exist and are continuous, but it is *analytic* meaning that in a neighbourhood of any point z_0 in U, the function f can be expressed as a *power series*

$$f(z) = f(z_0) + a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + \cdots$$

One starts by expressing a function f of one complex variable as the sum of two real valued functions in two real variables by the following isomorphism:

$$\begin{aligned} \mathbb{R} \times \mathbb{R} &\longrightarrow \mathbb{C} \\ (x, y) &\longmapsto x + iy \end{aligned} \tag{2}$$

It is then possible to prove that if f is a holomorphic map with non-zero derivative at a point z, then it is invertible in a neighbourhood of $f(z)$, and its inverse is holomorphic. This result is known as the *Inverse function theorem*.

A straightforward consequence of the definition of holomorphicity is that if

$$f(z) = f(x + iy) = u(x, y) + iv(x, y)$$

is holomorphic at $z_0 = x_0 + iy_0$ then the functions u and v are differentiable with continuous derivatives in both variables and satisfy the *Cauchy-Riemann equations* at (x_0, y_0):

$$\begin{cases} \partial_x u = \partial_y v \\ \partial_y u = -\partial_x v \end{cases} \tag{3}$$

This has two important consequences:

Theorem 2.1 (Cauchy Integral Theorem)**.** Let Ω be a region of $\mathbb{C}$ bounded by a simple closed curve and let $f : \Omega \to \mathbb{C}$ be holomorphic. Then the integral along a smooth curve γ in Ω is null:

$$\oint_\gamma f(z)\,\mathrm{d}z = 0 \tag{4}$$

Theorem 2.2 (Cauchy Integral Formula)**.** With the same setting as above, let z_0 be a point inside the domain bounded by γ. Then the value of f and its derivatives at z_0 are determined by the values on γ:

$$f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(\omega)}{\omega - z_0}\,\mathrm{d}\omega \tag{5}$$

One can prove using the Cauchy integral formula that if a function $f : U \to \mathbb{C}$ then it is analytic in a neighbourhood of any point z_0 in U with the following Taylor expansion:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

with

$$a_n = f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(\omega)}{(\omega - z_0)^{n+1}} \, \mathrm{d}\omega \tag{6}$$

Equation (6) is known as the Cauchy integral formula for derivatives. It implies further constraints to holomorphic functions.

Corollary 2.3 (Cauchy's Derivative Estimate)**.** If $f : \mathbb{D}_s(z_0) \longrightarrow \mathbb{D}_r(z_1)$ is holomorphic, then $|f'(z_0)| \leq r/s$.

Corollary 2.4 (Liouville's Theorem)**.** If a function $f(z)$ is bounded and holomorphic on the whole complex plane, then it is constant.

In the last sections of this work we will use functions in two complex variables. These are treated entirely analogously to functions in one complec variables. In particular the definition of holomorphic map $f : \mathbb{C}^2 \to \mathbb{C}$ is that the partial derivatives in both variables exist, and each satisfy the Cauchy-Riemann equations. The inverse function theorem works the same and gives rise to the implicit function theorem:

Theorem 2.5 (Implicit Function Theorem)**.** Let $U \subset \mathbb{C}^2$ be open set and let $f : U \to \mathbb{C}$ be holomorphic such that $f(z_0) = 0$ for some $z_0 = (x_0, y_0) \in U$. Then if $|\partial_x f(x, y_0)|_{x=x_0} \neq 0$, then there is a neighbourhood $V \subset \mathbb{C}$ of y_0 and a function $g : V \to \mathbb{C}$ and a neighbourhood $W \subset U \cap (V \times \mathbb{C})$ such that $f(x, y) = 0$ if and only if $y = g(x)$ for all $(x, y) \in W$.

Another consequence of the Cauchy integral formula is the Maximum Modulus principle:

Theorem 2.6 (Maximum Modulus Principle)**.** If f is holomorphic but not constant on a connected domain Ω then $|f|$ has no maximum inside Ω. The maximum is obtained on the boundary.

This theorem implies the following very useful lemma connecting geometry to analysis, that will be used time and again. It is useful to think of holomorphic functions as *maps*, sending subsets of the sphere to other subsets.

Lemma 2.7 (Schwarz Lemma)**.** If f is a holomorphic mapping of the unit disk $\mathbb{D}$ to itself that fixes the origin, then $|f'(0)| \leq 1$. Additionally, either

- $|f'(0)| = 1$ and f is a rotation: $f(z) = \lambda z$ with $\lambda \in \partial\mathbb{D}$, or
- $|f'(0)| < 1$ and f is a strict contraction: $|f(z)| < |z|$ for $z \neq 0$.

Proof. Define $q(z) = f(z)z^{-1}$. Then q is holomorphic on $\mathbb{D}$ and $q(0) = f'(0)$. Thus $q|_{\mathbb{D}_r}$ is also holomorphic, for all $r < 1$. Furthermore we have $\left| q|_{\mathbb{D}_r} \right| < |z^{-1}| < r^{-1}$. So for $z \in \mathbb{D}_r$ we have $q(z) \in \mathbb{D}_{1/r}$. By the Maximum Modulus Principle, $|q(z)|$ can only reach maximum on the boundary, so for all $z \in \mathbb{D}$ we have $q(z) \in \bar{\mathbb{D}}$.

- Suppose $|f'(0)| = 1$ then $|q(0)| = |f'(0)| = 1$. But we have seen that $|q(0)| \leq 1$ on $\mathbb{D}$ so by the Maximum Modulus Principle we have $q(z) = \lambda$ is a constant of modulus 1, thus f is a rotation.

- Suppose $|f'(0)| < 1$ then $|q(z)| < 1$ for all $z \in \mathbb{D}$ since if it equals section1 then by the Maximum Modulus Principle again $|f'(0)| = 1$, which is a contradiction. So $|f(z)| < |z|$ for all $z \in \mathbb{D}$ so f is a contraction. □

Definition 3. A holomorphic map $f : U \to \mathbb{C}$ is

- *conformal* if its derivative is nowhere 0 on U.
- *univalent* if it is injective.
- *biholomorphic* if it is bijective and conformal.

Another topological result s a consequence of the Cauchy Riemann equations is

Theorem 2.8 (Open Mapping Theorem)**.** The image of an open set by a holomorphic map is an open set.

2.2 The Riemann sphere and meromorphic functions

Liouville's Theorem seriously restricts which holomorphic functions are interesting. If f is a non-trivial holomorphic function of the entire complex plane, then it must obey

$$\lim_{z \to \infty} f(z) = \infty$$

Thus in complex analysis, one is prompted to deal with infinity. This is done by considering the *Riemann Sphere* $\hat{\mathbb{C}}$, the one point compactification of the complex plane. This is a prime example of a *Riemann surface*[1] other than $\mathbb{C}$ itself . If $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, then an atlas is provided by the two following charts:

$$\begin{aligned} \text{id } : \hat{\mathbb{C}} \backslash \{\infty\} &\longrightarrow \mathbb{C} \\ z &\longmapsto z \\[1em] \xi : \hat{\mathbb{C}} \backslash \{0\} &\longrightarrow \mathbb{C} \\ z &\longmapsto z^{-1} \\ \infty &\longmapsto 0 \end{aligned} \tag{7}$$

[1]a smooth complex manifold of one complex dimension

where $\xi(\infty) = 0$. Since the definition of a holomorphic function of a subset of $\mathbb{C}$ can be extended naturally to the sphere, by saying that a function f is holomorphic in a neighbourhood of infinity if the function

$$\tilde{f} = \xi \circ f \circ \xi^{-1}$$

is holomorphic in a neighbourhood of infinity. The function $\tilde{f}$ is called the *representative* f near infinity.

Working in the complex sphere allows us to consider functions that holomorphic except at a set of points called singularity. A function f has a singularity at z_0 if

$$\lim_{z \to z_0} f(z) = \infty$$

This is a *removable singularity* if there is a positive integer n such that $(z - z_0)^n f(z)$ does not have a singularity. This allows us to introduce another class of functions.

Definition 4. A function $f : U \to \hat{\mathbb{C}}$ is a *meromorphic* function if it is holomorphic and has a non-accumulating set of removable singularities, also called *poles*.

Ratios of two holomorphic functions are always holomorphic. In particular, all *rational functions*, that is functions of the form $f(z) = p(z)/q(z)$ where p and q are two polynomials with no common roots are meromorphic. Additionally, the only meromorphic functions of $\hat{\mathbb{C}}$ are the rational functions.

Proof. A rational function has a finite number of removable singularities: one at each of the roots of the denominator and one possibly at ∞ if the numerator has a higher degree than the denominator. Thus all rational functions are meromorphic. On the other hand, since the sphere is compact, every meromorphic function needs to have a finite number of singularities. Then if $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is meromorphic, so is $f|_{\mathbb{C}}$, which we can then multiply by a polynomial q to obtain a holomorphic function. Thus $z \mapsto f|_{\mathbb{C}}(z)q(z)$ is equal to its Taylor expansion. Moreover, for the singularity at infinity to be removable, the Taylor expansion must be of finite order, in other words, $f|_{\mathbb{C}}(z)q(z)$ is a polynomial. □

Proposition 2.9. Rational maps are surjective on the sphere.

Proof. Let $f(z) = p(z)/q(z)$. Then $f(z) = \infty$ once for every root of q. For any $\omega \neq \infty$ such that, the equation

$$\frac{p(z)}{q(z)} = \omega$$

is equivalent to

$$p(z) - \omega q(z) = 0$$

which by the fundamental theory of algebra, has $\deg f$ solutions counted with multiplicity. □

2.3 Normal Families and Montel's Theorem

We will be interested in a topology on meromorphic functions called the *topology of uniform convergence on compact subsets*, or *topology of local uniform convergence*. The *supremum*, or *uniform norm*, defined of a meromorphic function $f : U \to \mathbb{C}$ is given by

$$||f||_\infty = \sup_{x \in U} |f(x)|$$

A sequence $\{f_n : U \to \mathbb{C}\}$ of meromorphic functions is *uniformly convergent* if it converges under the uniform norm. Additionally it is said to converge to a function $f : U \to \mathbb{C}$ *locally uniformly* if for every compact subset K of U the sequence of maps $f_n|_K : K \to V$ converges uniformly to $f|_K$. Uniform convergence preserves holomorphicity.

Theorem 2.10 (Weierstrass Uniform Convergence)**.** The uniform convergence limit f of a sequence of holomorphic functions $f_n : U \to \mathbb{C}$ is also holomorphic. Furthermore, the sequence of derivatives f_n' converges uniformly on compact subsets to f'.

Proof. We first prove the uniform convergence of $\{f_n'\}$ to a holomorphic function g on compact subsets, then use that to show the convergence of $\{f_n\}$.

By assumption, $\{f_n\}$ is a Cauchy sequence in the supremum norm, so that $(f_n - f_m) : U \to \mathbb{D}_\epsilon$ for any $\epsilon > 0$ and n, m large enough. Additionally, for any $z \in U$, there is $r > 0$ such that $\mathbb{D}_r(z) \in U$ and we can apply the Cauchy derivative and get $|f_n'(z) - f_m'(z)| \leq \epsilon/r$. For each compact subset K of U there is a minimal radius, but ϵ can be made arbitrarily small by increasing n and m so $\{f_n'|_K\}$ is itself a Cauchy sequence. Additionally each f_n' is holomorphic on U so must be bounded on K. By the completeness of bounded continuous functions, $\{f_n'|_K\}$ converges uniformly to a continuous and bounded function g.

Now let γ be a path in U. Then $\int_\gamma f_n'(z)\mathrm{d}z \to \int_\gamma g(z)\mathrm{d}z$ as $n \to \infty$. Thus $f = \lim_{n\to\infty} f_n$ is an indefinite integral of g, which is a continuous function. Thus f is holomorphic and $g = f'$. □

Definition 5. A family $\mathcal{F}$ of functions $U \to \mathbb{C}$ is

- *bounded* if there is a $R > 0$ such that $||f||_\infty < R$ for all $f \in \mathcal{F}$,
- *equicontinuous* if for any z_0 in U and for any $\epsilon > 0$ there is a $\delta > 0$ such that if $|z - z_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$ for all $f \in \mathcal{F}$.

Lemma 2.11 (Ascoli-Arzelá)**.** Let K be a compact subset of $\mathbb{C}$. Any bounded and equicontinuous sequence of functions $f_n : K \to \mathbb{C}$ has a uniformly convergent subsequence.

Proof. Boundedness allows to construct a subsequence $\{f_{n_k}\}$ converging point-wise on a dense countable subset of K. The equicontinuity of the sequence and the compactness of K, we select a finite cover of K by disks of equal radius in which the variation from the centres is bounded. The fact that $\{f_{n_k}\}$ converges uniformly on at least a point in each of these disks is enough to prove that it converges uniformly on K. □

Theorem 2.12 (Ascoli-Arzelá)**.** A family $\mathcal{F}$ of functions $K \to \mathbb{C}$ is compact in the uniform convergence topology if and only if it is closed, bounded and equicontinuous.

Proof that I implies II. Suppose $\mathcal{F}$ is closed, bounded and equicontinuous. Then, every subsequence is bounded and equicontinuous, and thus has a uniformly convergent subsequence by Lemma 2.11. Additionally, since $\mathcal{F}$ is closed, the subsequence must converge to an element of $\mathcal{F}$, thus $\mathcal{F}$ is compact.

Suppose $\mathcal{F}$ is compact, then it is immediate that it be closed and bounded. Suppose $\mathcal{F}$ fails to be equicontinuous at some point z_0. Then there exists a sequence $\{z_n\}$ in K such that $|z_n - z_0| < 1/n$ and a sequence $\{f_n\}$ in $\mathcal{F}$ such that $|f_n(z_n) - f_n(z_0)| > \epsilon$ for some $\epsilon > 0$. However, since $\mathcal{F}$ is compact, there should be a subsequence f_{n_k} that converges uniformly on K to some continuous function f. In particular then $f_{n_k}(z_{n_k}) \to f_{n_k}(z_0)$ but this is a contradiction. □

Definition 6. A family $\mathcal{F}$ of meromorphic functions is *normal* if every sequence admits a subsequence that converges on compact subsets. A normal family is a pre-compact set under the topology of local uniform convergence.

Let us prove a criterion for a family $\mathcal{F}$ to be normal.

Lemma 2.13. If a family $\mathcal{F}$ of functions $U \to \mathbb{C}$ is equicontinuous on compact sets of U, then it is normal.

Proof. Since equicontinuous families are bounded on compact sets, $\mathcal{F}$ is bounded and equicontinuous on each compact set. Then considering an exhaustion of U by compact susbets on can use Lemma 2.11 and diagonal argument to select, for each sequence in $\mathcal{F}$ a subsequence that converges uniformly on all compact compact subsets. □

Theorem 2.14 (Little Montel Theorem)**.** Every bounded family $\mathcal{F}$ of holomorphic functions on an open subset U of $\mathbb{C}$ is normal.

Proof. The boundedness of $\mathcal{F}$ together with Cauchy's derivative estimate imply that the derivatives of the functions in $\mathcal{F}$ are bounded on compact subsets, which in turn implies that $\mathcal{F}$ is equicontinuous on compact subsets. Applying Lemma 2.13 completes the proof. □

This criterion can be strengthened by considering the following proposition.

Proposition 2.15 (Normality is a local property)**.** Let $\mathcal{F}$ be a family of functions $U \to \mathbb{C}$. If for every point in U, the restriction of the functions of $\mathcal{F}$ to an open neighbourhood form a normal family, then $\mathcal{F}$ is normal.

Proof. Let U_z denote the neighbourhood of each z in which the restrictions form a normal family. Each compact set can be covered by a finite number of such neighbourhoods and given any sequence in $\mathcal{F}$ one can use a diagonal argument to select a sequence that converges uniformly on K. Thus $\mathcal{F}|_K = \{f|_K \mid f \in \mathcal{F}\}$ is pre-compact and by (following a close argument to) the Ascoli-Arzelá Theorem, is bounded and equicontinuous. Lemma 2.13 in turn implies that $\mathcal{F}$ is normal. □

Corollary 2.16. A locally bounded family of meromorphic functions is normal.

Finally, we state an even less stringent condition for a family to be normal. This was shown in [9] by Montel in 1912. It is only thanks to this theorem that Fatou and Julia where able to make their drastic advances in the study of iteration [1]. We will see applications of this normality criterion below. The proof relies on a thorough study of holomorphic functions on hyperbolic Riemann surfaces, which is beyond the scope of this work. For a modern proof see Theorem 3.7 in [8].

Theorem 2.17 (Montel's Normality Criterion)**.** Let U be a domain of $\hat{\mathbb{C}}$, and let if $\mathcal{F}$ be a family of meromorphic functions $U \to \hat{\mathbb{C}}$ that omits three or more points of the sphere, *i.e.* there are three distinct points $a, b, c \in \hat{\mathbb{C}}$ such that for all f in $\mathcal{F}$, the points a, b, c are not in $f(U)$. Then $\mathcal{F}$ is normal.

3 Complex Dynamics

We start by introducing the basic concepts of dynamics, then move to introduce standard results. See [8] for a comprehensive exposition.

In dynamics, a meromorphic function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is thought of as a *map* telling each point in $\hat{\mathbb{C}}$ where to go. Given a map f, we denote called the n-th iterate the n-fold composition of f with itself, denoted by f^n. The set of points

$$\{x, f(x), f^2(x), f^3(x), \cdots\}$$

is called the *orbit* of x under f. If the orbit of a point x_0 is just $\{x_0\}$ call it a *fixed point*. The quantity $f'(x_0)$, called the *multiplier* and often denoted λ, plays a crucial role in determining the local dynamics. Fixed points are the solutions to the equation

$$f(x) = x$$

Alternatively, if p is the smallest positive integer satisfying $f^p(x_0) = x_0$, then x_0 is a *periodic* point and its orbit

$$\{x_0, x_1, x_2, \cdots, x_{p-1}\}$$

with $x_i = f^i(x_0)$ is called a *cycle*. The multiplier for this orbit is

$$(f^p)'(x_0) = f'(x_{p-1}) \times f'(x_{p-2}) \times \cdots \times f'(x_1) \times f'(x_0)$$

The classic objects of study in complex dynamics are the following two sets:

- The *Fatou set* $F(f)$ is the set of points z on which the family of iterates $\{f^n\}_{n\in\mathbb{N}}$, restricted to a neighbourhood of z, is normal.
- The *Julia set* $J(f)$ is the complement of the Fatou set.

The Fatou set is exactly the set where the dynamics are *Lyapunov stable*, meaning that points starting close to each other remain close for all iterations. More precisely, if z_0 is Lyapunov stable, for all $\epsilon > 0$ there is $\delta > 0$ such that if z is in $\mathbb{D}_\delta(z_0)$ then $f^n(z)$ is in $\mathbb{D}_\epsilon(f^n(z_0))$.

Proposition 3.1. A point is Lyapunov stable if and only if it is in Fatou set.

Proof. Saying that f is Lyapunov stable at z_0 is the dynamical way of saying that $\{f^n\}$ is equicontinuous at z_0. By the Ascoli-Arzelá theorem, whenever $\{f^n\}$ is normal, it is equicontinuous. Thus at all points in $F(f)$, f is Lyapunov stable. Assume now that z_0 is Lyapunov stable. Then if $\{f^n(z_0)\}$ is bounded away from ∞ then from equicontinuity, $\{f^n\}$ is bounded in a neighbourhood $U \ni z_0$ and thus normal by Corollary 2.16. If $\{f^n(z_0)\}$ is not bounded away from infinity, then there is a subsequence $\{f^{n_k}(z_0)\}$ converging to infinity. Looking at the subsequence $\{f^{n_k}\}$ in a neighbourhood of the point at infinity, we see that it is normal. □

However, except for the simplest maps, the dynamics are not everywhere stable.:

Lemma 3.2. Let f be a meromorphic function on the sphere, of degree at least 2. Then $J(f) \neq \emptyset$.

Proof. Argue by contradiction. Suppose $\{f^n\}$ is a normal family on $\hat{\mathbb{C}}$. Then there exists a subsequence $\{f^{n_j}\}$ that converges uniformly on compact subsets to a meromorphic function $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. This function must be a rational function of finite degree. But $\deg(f^{n_j})$ increases without bounds, which is a contradiction. □

Before moving on, we consider one of the simplest examples of complex dynamics:

Example. It is instructive to look at the dynamics of

$$f_0 : z \longmapsto z^2$$

because they can easily be treated analytically. Looking at the map in polar coordinates it is immediate that

$$\lim_{n\to\infty} f_0^n(z) = \begin{cases} 0 & \text{if } |z| < 1 \\ \infty & \text{if } |z| > 1 \end{cases}$$

So f^n converges point-wise to the constant function 0 on $\mathbb{D}$, but this convergence is not uniform. Indeed, since $|f_0^n(z)| = |z|^{2n}$, for any integer $n > 0$, $||f_0^n|_{\mathbb{D}}||_\infty = 1$. Thus uniform convergence on the whole domain fails to capture the behaviour of the orbits in $\mathbb{D}$. If we consider instead any compact subset K of $\mathbb{D}$, the situation is different. Indeed, since K is compact, there is $r = \max_{z\in K}|z|$ so that given any $\epsilon > 0$, for $|f_0^n(z)| < \epsilon$ for all $z \in K$ as soon as $n > \log(\epsilon)/\log(r)$. That is $||f_0^n||_K \to 0$ as $n \to \infty$ and $f_0^n|_{\mathbb{D}}$ converges locally uniformly to 0 and local uniform convergence captures the behaviour of iterates of $z \mapsto z^2$ inside the disk.

What about the behaviour for $|z| > 1$? Here we can use the coordinate chart near infinity. In this chart, f becomes

$$\tilde{f}_0 : \omega \mapsto f_0(\omega^{-1})^{-1} = \omega^2$$

and thus the behaviour is the same as that around 0. Thus $\{f_0^n\}$ is normal on $\mathbb{D}$ and on $\hat{\mathbb{C}}\backslash\bar{\mathbb{D}}$. Thus $\hat{\mathbb{C}}\backslash\partial\mathbb{D} \subset F(f_0)$. Indeed one can see that $\hat{\mathbb{C}}\backslash\partial\mathbb{D} = F(f_0)$ since no subsequence of $\{f_0^n\}$ can ever converge to a holomorphic function on a neighbourhood of a point in $\partial\mathbb{D}$, arbitrarily close points inside and outside $\mathbb{D}$ converging to different points.

Thus $J(f) = \partial\mathbb{D}$. The since $f_0 : e^{i\theta} \mapsto e^{2i\theta}$, by writing $\theta = 2\pi x$, the dynamics here are conjugate to the doubling map on the unit torus:

$$x \longmapsto 2x \mod 1$$

which is a prime example of a simple chaotic map. All points such that $x = 2^{-n}$ are eventually mapped to $z = 1$, which is a fixed point of this map. For odd p, the points with $x = k/p$, that is the p-th roots of unity, form a cycle, and that points with $x = k/(2^n p)$ are eventually mapped to this cycle. This characterises the behaviour of all points with x rational. It is possible to show that the orbit of any point for which x is irrational is dense on $\partial\mathbb{D}$. □

Proposition 3.3. The Fatou set is open, the Julia set is closed.

Proof. The Fatou set is open because of the local nature of normality. The Julia set is closed since it is its complement. □

A set U is *fully invariant* under a map f if $U = f(U) = f^{-1}(U)$. The following results simplify the study of periodic orbits, since they allow to look at the dynamics of iterates of f, where the cycle is split into fixed points.

Lemma 3.4 (Invariance Lemma)**.** The Fatou and Julia sets are fully invariant.

Proof. That $F(f)$ is fully invariant follows from the fact that holomorphic maps are continuous, open maps. $J(f)$ is then fully invariant too. □

Lemma 3.5 (Iteration Lemma)**.** The Fatou and Julia set of f are the same as those of any iterate f^k.

Proof. It suffices to prove that $F(f^k) = F(f)$. If $\{f^n\}$, then so is $\{f^{nk}\}$, thus $F(f) \subset F(f^k)$. Then consider that if $\{f^{nk}\}$ is normal, so is $\{f^i \circ f^{nk}\}$ for any $i = 1, 2, \cdots, n-1$, so that $\{f^n\} = \bigcup_{i=1}^{n-1} \{f^i \circ f^{nk}\}$ is normal and $F(f^k) \subset F(f)$. □

3.1 Local Fixed Point Theory

A number of things can be deduced about the dynamics of the mapping in the neighbourhood of fixed or periodic points. Indeed the local dynamics at a point $z = f^p(z)$ will be determined by the *multiplier* $\lambda = (f^p)'(z)$. In the following discussion, we will often set $p = 1$ and assume without loss of generality that $z = 0$ is the fixed point, so that

$$f(z) = \lambda z + a_2 z^2 + a_3 z^3 + \cdots$$

A fixed point z is *topologically attractive* if there exists a neighbourhood U of z such that successive iterates f^n are defined on U and that $f^n \to z$ uniformly on U. A fixed point z is *topologically repelling* if there exists a neighbourhood V of z such that for all x in V other than z there is a positive integer n such that $f^n(x) \notin V$. There is a simple test to determine if a fixed point is attracting or repelling.

Proposition 3.6. A fixed point of a holomorphic map is topologically attractive if and only if its multiplier satisfies $|\lambda| < 1$.

Proof. We assume that $f(0) = 0$ as above so that there are constants $C, r_0 > 0$ such that

$$|f(z) - \lambda z| < C|z^2|$$

for all $|z| < r_1$. Then we can choose c such that $|\lambda| < c < 1$, and there is $r_2 > 0$ such that $|\lambda| + Cr_1 < c$. Then for all $|z| < r_0 = \min\{r, r_0\}$,

$$|f(z)| \leq |\lambda z| + C|z^2| \leq c|z| < |z|$$

so that upon iteration we have for all $z \in \mathbb{D}_{r_0}$

$$f^n(z) \leq c^n|z| \leq c^n|r_0|$$

meaning that $f^n(z) \to 0$ uniformly over $\mathbb{D}_{r_0}$.

Conversely, if 0 is topologically attracting, then an iterate f^n will map a sufficiently small disk $\mathbb{D}_\epsilon$ onto a proper subset of itself, and by the Schwarz lemma, this will imply $|\lambda^n| = |(f^n)'(0)| < 1$ which is equivalent to $|\lambda| < 1$. □

Proposition 3.7. A fixed point of a holomorphic map is topologically repelling if and only if its multiplier satisfies $|\lambda| > 1$.

Proof. That a fixed point with multiplier $|\lambda| > 1$ is topologically repelling is very similar to the proof of the case $|\lambda| < 1$.

To prove the converse, assume that z_0 is a repelling fixed point. Then z_0 cannot be topologically attracting and, by the previous result, $\lambda \geq 1$. Let V be the neighbourhood such that for all z in V there is a positive integer n such that $f^n(z) \notin V$. Then for each $k \in \mathbb{N}$ we can construct the set

$$V_k = N \cap f^{-1}(V) \cap f^{-2}(V) \cap \cdots \cap f^{-k}(V)$$

of points whose first k images are in V. By construction $f(V_k) \subset V_{k-1} \cap f(V)$. However $\{V_k\}_{k \in \mathbb{N}}$ is a sequence of nested sets whose intersection is the set $\{z_0\}$ consisting of the single point that does not escape V, we have diam $V_k \to 0$ as $k \to \infty$. Therefore $V_{k-1} \subset g(V)$ and $f(V_k) \subset V_{k-1}$ for k large enough. Then one can use the Schwarz lemma to prove that $|\lambda| > 1$. □

Definition 7. A fixed point is *superattracting* or *geometrically attracting* if $\lambda = 0$ or $0 < |\lambda| < 1$, respectively.

Theorem 3.8. *Kœnigs Linearisation* Let f be a holomorphic map with periodic point z_0 of period p and multiplier λ such that $|\lambda| \neq 0, 1$. Then there exists a unique biholomoprhic mapping ϕ_{z_0} from a neighbourhood of z_0 to a neighbourhood of 0 such that

$$\phi_{z_0} \circ f^p(z) = \lambda \phi_{z_0}(z)$$

such that $(\phi_{z_0})'(z_0) = 1$.

Proof for attracting case. Proof of uniqueness. Let 0 be the fixed point. Assume there is another such map ψ. If we write $\omega = \phi(z)$ then the transition function $\psi \circ \phi^{-1}$ is holomorphic, with Taylor expansion:

$$\psi \circ \phi^{-1}(\omega) = b_1\omega + b_2\omega^2 + b_3\omega^3 + \cdots$$

Additionally, this map commutes with multiplication by λ since

$$\psi \circ \phi^{-1}(\lambda\omega) = \psi \circ \phi^{-1}\Big(\phi \circ f(z)\Big) = \psi \circ f(z) = \lambda\psi(z) = \lambda\psi \circ \phi^{-1}$$

Thus we see that for all $n \geq 1$ we have $\lambda^{n-1}b_n = b_n$ and since λ is neither 0 nor a root of unity, we must have $b_n = 0$ for all $n \geq 2$. It follows that $\psi(z) = b_1\phi(z)$, so that the two functions differ only by an overall scaling. By requiring $\phi'(0) = 1$ one chooses a unique map.

Proof of existence. Same setting as in the proof of Lemma 3.6, with the added constraint that c is chosen so that it satisfies $c^2 < |\lambda| < c < 1$. Then for all z in $\mathbb{D}_{r_0}$,

$$|f^{n+1}(z) - \lambda f^n(z)| \leq C|f^n(z)|^2 < Cr_0^2c^{2n}$$

Dividing both sides by $|\lambda^{n+1}|$ and setting $\phi_n(z) = \lambda^{-n}f^n(z)$ yields that $|\phi_{n+1}(z) - \phi_n(z)|$ converges to 0 uniformly and geometrically so that the functions ϕ_n converge uniformly to a holomorphic function ϕ. The identity $\phi \circ f(z) = \lambda\phi(z)$ follows, as well as $\phi'(0) = 1$, so that this is the required conformal map. □

Proof for repelling case. The point z_0 is a fixed repelling point of f^p with multiplier $\lambda \neq 0$. Then there are two neighbourhoods $U \supset V \ni z_0$ such that $f^p|_V : V \to U$ is biholomorphic, with inverse $g : U \to V$. The point z_0 is then an attracting fixed point of g with multiplier λ^{-1}. We then obtain the Kœnigs linearisation ϕ_{z_0} for g. We can assume that U was chosen small enough that ϕ_{z_0} is well-defined on the whole of U. Then for all $\omega \in U$ we have

$$\phi_{z_0} \circ g(\omega) = \lambda^{-1}\phi_{z_0}(\omega)$$

Then setting $z = g(\omega) \iff \omega = g^{-1}(z) = f^p(z)$, we have, for all $z \in V$:

$$\begin{aligned} \phi_{z_0}(z) &= \lambda^{-1}\phi_{z_0} \circ g^{-1}(z) \\ \iff \lambda\phi_{z_0}(z) &= \phi_{z_0} \circ f^p(z) \end{aligned} \tag{8}$$

□

Kœnigs linearisation characterises the behaviour of a map in the neighbourhood of repelling periodic points attracting periodic points with multipliers $|\lambda| \neq 0, 1$ and justifies calling them *geometrically* repelling, attracting points. The points with multiplier $\lambda = 0$ are called *superattracting*. Let z_0 be a superattracting fixed point of a function f. We call the *local degree* of f at z_0 the first integer n such that $(f^n)'(z_0) \neq 0$. The following result justifies the term superattracting.

Theorem 3.9 (Böttcher Theorem)**.** Let z_0 be a superattracting fixed point of a map f. Then there exists a unique biholomoprhic mapping ϕ from a neighbourhood of z_0 to a neighbourhood of 0 such that

$$\phi \circ f(z) = \phi(z)^n$$

where n is the local degree of f at z_0. This map is unique up to multiplication by a $(n-1)th$ root of unity.

The proof of this function theorem is along the lines of that of Kœnigs linearisation. To prove existence one considers the convergence of the functions

$$\phi_k(z) = f^k(z)^{1/n^k}$$

for a particular choice of the branch of the root.

Definition 8. The *basin of attraction* $\mathcal{A} = \mathcal{A}(z_0)$ of an attracting point z_0 is the set of all points z such that $f^n(z) \to z_0$ as $n \to \infty$. The *immediate basin* $\mathcal{A}_0$ is the connected component of $\mathcal{A}$ that contains z_0.

From the foregoing, we can see that attracting and superattracting fixed points of f, as well as their basins of attraction, belong to $F(f)$, since the iterates converge locally uniformly to the fixed point. The same can be said for attracting and superattracting periodic points, since $F(f^p) = F(f)$. Additionally, we can see that repelling periodic and fixed points belong to $J(f)$. This is because the derivative of f^n grows geometrically with n, no sequence of iterates can converge to a meromorphic function.

We have fully characterised the behaviour around periodic points with multiplier $|\lambda| \neq 1$. The behaviour at these points, deemed *indifferent* or *neutral* is more subtle and depends on the algebraic properties of the multiplier. If λ is a root of unity, then the fixed point is called *parabolic*. Assume 0 is a parabolic fixed point then there is q such that $\lambda^q = 1$ and

$$f^q(z) = z + a_m z^m + \cdots$$

so that iterates f^{nq} can never converge since their p-th derivative at 0 grows arithmetically with n. Thus parabolic points belong to the Julia set. However, it is possible to show that in any neighbourhood U of 0 there are points that converge uniformly to 0.

For further discussion, see [8].

3.2 Properties of the Julia set

These are standard results, mostly consequences of Montel's normality criterion (Theorem 2.17). These results were pioneered by Gaston Julia and Pierre Fatou before the advent of computer technology to visualise the stunning structures it describes.

We call *the grand orbit* of a point z_0 in $\hat{\mathbb{C}}$ the set of points whose orbit eventually intersects that of z_0. In other words, the grand orbit of z_0 is the set of points z for wich there are n and m such that $f^n(z_0) = f^m(z)$.

Proposition 3.10. Let f be rational function of degree at least 2. Then there are at most two points whose grand orbit is finite, and they must be superattracting periodic points.

Proof. Let G be the grand orbit of z. Clearly G is fully invariant, meaning that $f(G) = G = f^{-1}(G)$ and so if G has finite cardinality, f maps G bijectively onto itself, thus G is a cycle. Additionally, since every point has deg $f \geq 2$ pre-images counted with multiplicity, each point in G is a critical point. Suppose now that there are three points $a, b, c \in \hat{\mathbb{C}}$ whose grand orbit is finite. Then f maps subsets of $\hat{\mathbb{C}} \backslash \{a, b, c\}$ to $\hat{\mathbb{C}} \backslash \{a, b, c\}$, and thus the family of iterates is normal everywhere on $\mathbb{C}$ by Montel's Normality Criterion. This is a contradiction since $J(f) \neq 0$. □

This implies that the dynamics of f in any neighbourhood of a point on the Julia set are *transitive*. This has a number of interesting consequences on the dynamics and structure of the Julia set. Transitivity is taken as one of the signs that a system is chaotic.

Theorem 3.11 (Transitivity)**.** Let $z_0 \in J(f)$ and let N be an arbitrary neighbourhood of z_0. Then its *orbit*

$$U = \bigcup_{n=0}^{\infty} f^n(N)$$

contains all of the Julia set, and all but at most two points of $\hat{\mathbb{C}}$.

Proof. Since $z_0 \in J(f)$, $\{f^n\}$ is not normal in any neighbourhood of z_0, thus $\hat{\mathbb{C}} \backslash U$ must consist of at most two points a, b by Montel's normality criterion. Since the a has no pre-image in U, but must have at least one preimage, it is either fixed a fixed point or part of a two-cycle with b. Thus a and b have finite grand orbit is finite and hence belong to $F(f)$. All the points in $J(f)$ are then in U. If N is chosen small enough to not contain neither of the finite grand orbit points, then $\hat{\mathbb{C}} \backslash U$ will consist exactly of the set of points whose grand orbits are finite. □

This has the following consequence for the dynamics of f on $J(f)$:

Corollary 3.12 (Iterated preimages are dense)**.** The set of pre-images $f^{-n}(z_0)$, $n \in \mathbb{N}$, of z_0 is dense in $J(f)$.

Proof. Consider a point $z \in J(f)$ and any neighbourhood N of it. Since by transitivity the orbit of N contains all $J(f)$, there exists at least one point ω in N such that $f^m(\omega) = z_0$, for some m. Since $J(f)$ is invariant, $\omega \in J(f)$. □

Perhaps more interesting are the consequences on the structure of the Julia set itself.

Corollary 3.13. Either $J(f)$ has no interior, or $J(f) = \hat{\mathbb{C}}$.

Proof. Assume z_0 is in the interior of $J(f)$. Then there is a neighbourhood $N \subset J(f)$ of z_0, and by invariance of $J(f)$, the orbit U of N is contained in $J(f)$. However by transitivity, U omits at most two points, and so does $J(f)$. Since $J(f)$ is closed $J(f) = \hat{\mathbb{C}}$. □

Corollary 3.14. $J(f)$ has no isolated points.

Proof. If $J(f)$ were finite, by invariance it would be a finite grand orbit, hence in $F(f)$. So $J(f)$ is infinite. Since $J(f)$ is compact, it contains at least one accumulation point z_0. The pre-image of a sequence accumulating at z_0 contains sequences accumulating at the pre-images of z_0. Since the iterated pre-images of z_0 are dense in $J(f)$ we have a dense set of accumulation points, so no point is isolated. □

Corollary 3.15. Let $\mathcal{A}$ be a basin of attraction of an attracting periodic point. Then $J(f) = \partial\mathcal{A}$.

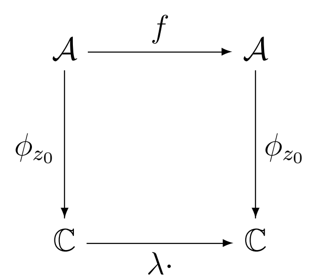
Proof. Let $z_0 \in J(f)$. By transitivity the orbit of any neighbourhood N of z_0 intersect $\mathcal{A}$, which in turn implies that N intersects $\mathcal{A}$ and $z_0 \in \partial\mathcal{A}$. Now assume $z_0 \in \partial\mathcal{A}$ and let N be a neighbourhood of z_0. If p is the period of the attracting cycle, any limit of $\{f^{pn}|_N\}$ will have a discontinuity. Thus $z_0 \in J(f)$. □

This last result has profound implications on the structure of the Julia set whenever there are more than two attractors, as Hubbard noticed when looking at the behaviour of Newton's method for a third order polynomial. Indeed, a set has to be very complicated if every point has to be on the boundary of three sets. The reader is invited to try and draw what a set like this would look like.

3.3 Global Theory

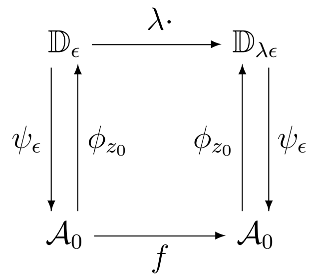
In this section, we present a final characterisation of the structure of the Fatou and Julia sets. This will culminate in the statement of the classification of periodic Fatou components.

Lemma 3.16. The Kœnigs linearisation ϕ_{z_0} of a geometrically attracting fixed point extends to the whole of the basin of attraction, so that the following diagram commutes.



Proof. The Kœnigs linearisation ϕ_{z_0} is defined on a neighbourhood U of z_0. The orbit of every point in $\mathcal{A}$ will eventually enter U, that is, for any z in $\mathcal{A}$ there is n such that $f^n(z)$ is in U. We then simply set $\phi_{z_0}(z) = \lambda^{-n}\phi_{z_0}\big(f^n(z)\big)$. □

Since $\phi'_{z_0}(z_0) = 1$, it maps a neighbourhood of z_0 biholomorphically to a neighbourhood of 0. This means that there is $\epsilon > 0$ a conformal map $\psi_\epsilon : \mathbb{D}_\epsilon \to \mathcal{A}_0$ such that the following diagram commutes:



However, unlike ϕ_{z_0} this cannot be extended to the whole of $\mathbb{C}$. Indeed there is a maximal value for r.

Lemma 3.17. The local inverse ψ_ϵ of the Koenig linearisation around an attracting fixed point extend to some maximal open disk $\mathbb{D}_r$. It extends homeomorphically to $\partial\mathbb{D}_r$ and the image of this circle contains a critical point of f.

Proof. Step 1. There exists some maximal r such that the inverse can be extended analytically. Assume this is not the case. Then the inverse can be extended to a function over the entire complex plane. If $\mathcal{A}_0$ is bounded, then Liouville's theorem implies that ψ_ϵ is constant. If $\mathcal{A}_0$ is not bounded, then the image would still omit all the points in the Julia set. Since $J(f)$ consists of infinite points, by Picard's theorem[2] the inverse is constant. This is a contradiction so there must be a maximal r.

Step 2. The function is well defined and extend homeomorphically to the boundary. By step 1, there exists a maximal $r > 0$ such that there exists a local inverse $\phi^{-1} : \mathbb{D}_r \to \mathcal{A}_0$. Now take any ω on $\partial\mathbb{D}_r$. Then $\lambda\omega$ is in $\partial\mathbb{D}_r$. Now write $\phi^{-1}(\omega) = f^{-1} \circ \phi^{-1}(\lambda\omega)$, where f^{-1} is the local inverse of f on $\mathcal{A}_0$, which exists by definition of r.

Step 3. $\phi^{-1}(\partial\mathbb{D}_r)$ contains a critical point of f. Consider $\omega_0 \in \partial\mathbb{D}_r$. If $\phi^{-1}(\omega_0)$ is not a critical point of f, then there exists a local inverse g of f around $\phi^{-1}(\omega_0)$. Now there are values of ω close to ω_0 for which $\phi(\lambda\omega)$ lands in the neighbourhood where g is defined, so that one define $\phi^{-1}(\omega) = g \circ \phi^{-1}(\lambda\omega)$, thus extending the domain of ϕ^{-1}. Since $\partial\mathbb{D}_r$ is a compact set, if there were no $\omega_0 \in \partial\mathbb{D}_r$ such that $\phi^{-1}(\omega_0)$ is critical, one could extend ϕ^{-1} to a larger disk $\mathbb{D}_{r'}$, which contradicts step 1. So there must be at least one critical point on $\phi^{-1}(\partial\mathbb{D}_r)$. □

[2]This a stronger version of Liouville's theorem. It says that if a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ misses two values, then f is constant. Like Montel's normality condition, it is result from the study of hyperbolic Riemann surfaces, which is beyond the scope of this work. In the case of a polynomial function, the basin of a geometrically attracting periodic cycle will always be bounded.

Theorem 3.18. For any rational map f of degree at least 2, there is a critical point in the immediate boundary of attraction to any periodic attracting cycle.

Proof. Superattracting cycles contain critical points so the immediate basin contains a critical point. If $z_0 \mapsto z_1 \mapsto \cdots \mapsto z_p = z_0$ is a geometrically attracting cycle, then z_0 is a fixed point of f^p. By the previous lemma, there is a critical point c of f^p in the immediate basin of z_0. But since $Df^p(c) = f'(c)f'\Big(f(c)\Big)\cdots f'\Big(f^{p-1}(c)\Big)$ one of the points in the orbit of c under f must be a critical point of f. Thus there is a critical point in the immediate basin of the cycle starting at z_0. □

With a similar albeit more careful analysis, one can prove an analogous result for parabolic periodic points:

Theorem 3.19. Every parabolic basin of attraction contains a critical point.

Together, these result on an upper bound on the number of attracting and parabolic periodic points.

Theorem 3.20 (Number of attracting and parabolic cycles)**.** The number of attracting periodic cycles plus the number parabolic cycles is equal to the number of distinct critical points. This number for a rational map of degree $d \geq 2$ is at most $2d - 2$.

We state the following powerful results, and give some details.

Theorem 3.21 (Sullivan's Classification of Fatou Components)**.** If a rational map of degree at least 2 maps a connected component U to itself, then there are only four possibilities. Either U is the immediate basin of attraction of an attracting fixed point, or a parabolic fixed point, or U is a *Siegel disk* or *Herman ring*. The last two are domains associated with irrational indifferent points, they are topologically a disk and an annulus respectively, where the dynamics are conjugate to an irrational rotation.

Siegel disks and Herman rings bear the name of the authors that proved their existence, Carl Siegel in 1942 [16] and Michael Herman in 1979 [17]. Sullivan then proved in [18] that attracting and parabolic basins and the two rotation domains are the only four possibilities for invariant Fatou components of rational function. He also proved that all Fatou components are eventually mapped to a periodic Fatou component, which is known as the *Sullivan Nonwandering Theorem*. Shishikura proved in [19] that there can be at most $2d - 2$ distinct cycles of Fatou components.

This in turn implies that most periodic orbits are repelling and in fact,

Theorem 3.22. The set of repelling periodic orbits is dense in the Julia set.

4 The quadratic family

The *quadratic family* is the one parameter set of functions

$$f_c : z \mapsto z^2 + c$$

where c is complex. This is the simplest non-trivial family of rational functions, however their dynamics are still subject of intense study. Studying the quadratic family, on learns about the dynamics of all quadratic polynomials. Indeed if one conjugates f_c by an affine transformation $z \mapsto az + b$ one obtains

$$\omega \mapsto a\left[\left(\frac{\omega - b}{a}\right)^2 + c\right] + b = \frac{1}{a}\omega^2 - \frac{2}{a}b\omega + (ac + b + \frac{1}{a}b^2)$$

which, by selecting a, b and c properly can be made into any quadratic polynomial.

For every c, the point at infinity is a superattracting fixed point. To see this, look at the map in a neighbourhood of infinity:

$$\tilde{f}_c(\omega) = \frac{1}{\omega^{-2} + c} = \frac{\omega^2}{1 + \omega^2 c}$$

We see that 0 is indeed a superattracting fixed point of $\tilde{f}_c$.

This motivates considering the set of points whose orbits are bounded. This is K_c, *filled Julia set*. One immediate result is that K_c is contained in the disk closed disk of radius $b(c) = \max\{2, \sqrt{2|c|}, \sqrt{|c|}\}$ centred at the origin. Indeed if $|z| > b(c)$ then we have

$$\begin{aligned} |f_c(z)| = |z^2 + c| \geq |z^2| - |c| = |z|^2\left(1 - \frac{|c|}{|z^2|}\right) \\ > \frac{1}{2}b(c)|z| \geq |z| \end{aligned} \tag{9}$$

So that iterates f_c^n converges uniformly to infinity on the complement of any disk of radius greater than $b(c)$.

Proposition 4.1. The filled Julia set K_c is a compact subset of $\mathbb{C}$, with connected complement. Its boundary is the Julia set $J_c = J(f_c)$ and its interior is the union of the bounded components of $F_c = F(f_c)$.

Proof. Evidently, K_c is the complement of the basin of attraction $\mathcal{A}$ of the point at infinity, so it is a closed set. Additionally, since the immediate basin of infinity is an open neighbourhood of infinity, K_c is also bounded, proving that K_c is compact. It also follows that $\delta K_c = J_c$. Finally, to prove that $\mathcal{A}$ is connected, we prove that any bounded component U of F_c is not a component of $\mathcal{A}$. Suppose U is a component of $\mathcal{A}$, then for a point $z_0 \in U$ there is n such that $f_c^n(z_0) > b(c)$. Then the Maximum Modulus implies that there is a point z on $J_c = \partial U$ such that $f_c^n(z) > b(c)$ and thus $z \in \mathcal{A}$, which is a contradiction. □

Proposition 4.2. The bounded Fatou components of f_c are simply connected.

Proof. Let U be a bounded component of $F(f_c)$ and consider a simple closed curve γ and the set V that it bounds. V must be a subset of K, since otherwise there would be a point z_0 such that $f_c^n(z_0) > b(c)$ and by the Maximum Modulus Principle, there would be a point z on $\gamma \subset U$ such that $f_c^n(z) > b(c)$, and thus $z \in \mathcal{A}_\infty$, which is a contradiction. Additionally, since $J_c = \partial \mathcal{A}_\infty$, there can be no point in V that is also in J_c. Thus $V \subset U$. □

This implies that there are no Herman rings. Additionally, 0 and ∞ are the only critical points, there can only be one Fatou cycle, and it must either consist of the basin of an attracting or parabolic cycle, or of Siegel disks.

Theorem 4.3 (Standard Dichotomy)**.** Either 0 is in K_c and K_c is connected, or 0 is attracted to infinity and K_c has uncountably many components.

We finally arrive at the definition of the Mandelbrot set:

Definition 9. The Mandelbrot set M is the set of parameters c such that K_c is connected.

By Theorem 4.3 this definition, and the one in the introduction are equivalent. However, this one carries the conceptual weight of complex dynamics.

For example, when one looks at computer generated pictures of the Mandelbrot set, there are "islands" that appear, resembling the whole set. When Mandelbrot first saw these pictures, he conjecture that M was disconnected. One can in fact prove that the Mandelbrot set is actually simply connected. This is done by looking at the function:

$$\begin{aligned} B : \hat{\mathbb{C}} \backslash M &\longrightarrow \hat{\mathbb{C}} \backslash \bar{\mathbb{D}} \\ z &\longmapsto B_c(c) \end{aligned} \tag{10}$$

where B_c corresponds to the Böttcher function for f_c. It can be proven in a way analogous to the proof of Proposition 8.1 below, that this is a holomorphic map, tangent to the identity at infinity and that it is *proper*: mapping the boundary of M to the boundary of the disk.

Another interesting result by Mitsuhiro Shishikura [20] is that in a neighbourhood a generic point on the boundary of M, the Hausdorff dimension of ∂M is 2.

A current open question is whether the boundary of M is locally connected. See [11].

4.1 Misiurewicz Parameters

Lemma 4.4. If c is Misiurewicz, then c cannot be periodic.

Proof. If c is periodic, then there is p such that $c = f_c^p(c)$ but since $f_c^p(c) = f_c^{p+1}(0) = f_c^p(0)^2 + c$, this means that $f_c^p(0)$ must be 0, so that 0 is periodic. c cannot be Misiurewicz. □

Theorem 4.5. If c is Misiurewicz, then $K_c = J_c$.

Proof. Let $f_c^p \circ f_c^l(0) = f_c^l(0)$. $\alpha = f_c^l(0)$ is a fixed point of $g = f_c^p$ and by the Invariance Lemma, $K(g) = K_c$. So it will suffice to look at which kind of fixed point α is. Notice that all the critical points of g are pre-images of 0 under f_c. This means that the multiplier of the orbit is not 0. Also, since the critical point is a pre-image under g of the eventually fixed point, it cannot lie in the basin of a geometrically attracting, superattracting or parabolic point. There can be no Siegel disk, as this requires the accumulation of a post-critical orbit to its boundary, and the orbit of the critical points is finite so it cannot accumulate. Finally, there can be no Herman ring because the bounded Fatou components must be simply connected. □

Corollary 4.6. If c is Misiurewicz, then all periodic orbits are repelling.

Corollary 4.7. If c is Misiurewicz, then 0 and c are eventually repelling periodic.

5 Self-Similarity

In order to talk about self-similarity, we need to formalise the concept of two shapes looking like each other, or for a shape to look like itself. These are notions of equality and convergence, and hence topology is the natural setting. As the filled Julia sets are compact subsets of $\mathbb{C}$, we will use the Euclidean norm on the complex plane to define a metric topology on the space of compact subsets of $\mathbb{C}$.

5.1 Definitions

Definition 10. Let H denote the set of non-empty compact subsets of $\mathbb{C}$. Let $z \in \mathbb{C}$ and $A, B \in H$.

- The distance of z to B is defined to be the smallest distance of z to an element of B:

$$d(z, B) = \min_{b \in B} |z - b| \tag{11}$$

- We define the *semidistance* of A to B to be the highest distance of an element of A to the set B:
$$\delta(A,B) = \max_{a\in A} d(a,B) \tag{12}$$

- This in turn lets us define the *Hausdorff distance* between two compact sets:
$$d_H(A,B) = \max\Big\{\delta(A,B), \delta(B,A)\Big\} \tag{13}$$

To motivate d_H, one can notice that the semidistane fails to be a metric. Although it is non-negative, it is easy to see that $\delta(A,B) = 0$ if and only if A is a subset of B. Additionally, $\delta(A,B) \neq \delta(B,A)$. However it is easy to check that it does satisfy

$$\delta(A,C) \leq \delta(A,B) + \delta(B,C)$$

We can look at what a neighbourhood looks like under the semidistance. Indeed $\delta(A,B) \leq \epsilon$ if an only if any point in A is either in B or within ϵ from it. Let

$$N_\epsilon(B) = \{c \in \mathbb{C} \mid d(c,B) \leq \epsilon\}$$

then $\delta(A,B) \leq \epsilon$ if and only if A is a subset of $N_\epsilon(B)$.

Proposition 5.1. The space (H, d_H) is a complete metric space.

Proof. To prove that d_H is a metric, consider:

- *Positive definiteness*. Non-negative by definition and since $\delta(A,B) = 0$ if and only if $A \subset B$, $d_H(A,B) = 0$ if and only if $A = B$.
- *Symmetry*. By definition.
- *Triangle Inequality*. By definition.

The proof of completeness is longer, see [4]. □

In the analysis of (self-)similarity, we often look in the neighbourhood of specific points: we are interested in the sets within a certain window.

Definition 11. The *Hausdorff-Chabauty distance in the window of* $\mathbb{D}_r$ is defined for any closed subsets A and B of $\mathbb{C}$ as follows:

$$d_r(A,B) = d_H\Big(\ (A)_r, (B)_r\Big) \tag{14}$$

where

$$(A)_r = (A \cap \mathbb{D}_r) \cup \partial\mathbb{D}_r$$
$$(B)_r = (B \cap \mathbb{D}_r) \cup \partial\mathbb{D}_r$$

Note that the union with the circle is taken to ensure compactness of the resulting sets. Indeed to prove $(A)_r = (B)_r$ it is sufficient to prove $A \cap \mathbb{D}_r = B \cap \mathbb{D}_r$ *i.e.* that they overlap on the open disk. As long as A is closed, $(A)_r$ is compact.

Proposition 5.2. Triangle inequalities and relations for d_r d_H and δ. For any $z \in \mathbb{C}$ and $A, B \in H$.

1. $d(z, A) \leq d(z, B) + d_H(A, B)$
2. $\delta\Big((A)_r, (B)_r\Big) = \delta\Big(A, (B)_r\Big)$
3. $d(z, (A)_r) \leq r$ for any $z \in \bar{\mathbb{D}}_r$
4. $d(z, (A)_r) \leq d(z, A)$ for any $z \in \bar{\mathbb{D}}_r$
5. $d_r(A, B) = d_H(A, B)$ as soon as $A, B \subset \bar{\mathbb{D}}_r$

Proof. 1.

$$\begin{aligned} d(z, A) &= \min_{a\in A} |z - a| \\ &\leq \min_{b\in B} \min_{a\in A} |z - b| + |b - a| \\ &\leq \min_{b\in B} |z - b| + \min_{b\in B} \min_{a\in A} |b - a| \\ &\leq d(z, b) + \min_{b\in B} d(b, A) \\ &\leq d(z, b) + d_H(A, B) \end{aligned} \tag{15}$$

2. This true, as the distance of points in $\partial\mathbb{D}_r$ to $(B)_r$ is identically 0.
3. On the one hand $d\Big(z, (A)_r\Big)$ is either the distance from z to $A \cap \bar{\mathbb{D}}_r$ or $r - |z|$, the distance from the circle. On the other hand, $d(z, A)$ is either the distance from z to $A \cap \bar{\mathbb{D}}_r$, or the distance from z to the part of A beyond the circle, which is always greater than $r - |z|$. So $d\Big(z, (A)_r\Big) \leq d(z, A)$.
4. and 5. are self-evident.

□

We are now ready to formalise notions of (self-)similarity.

Definition 12. Let $\rho \in \mathbb{C}\backslash\bar{\mathbb{D}}$ and A and B two closed subsets of $\mathbb{C}$.

- A is *ρ-self-similar* about 0 if for some $r > 0$

$$(A)_r = (\rho A)_r$$

Similarly, A is ρ-self-similar *about* x if for some $r > 0$

$$(\tau_{-x}A)_r = (\rho\tau_{-x}A)_r$$

where $\tau_{-x} : z \mapsto z - x$ is a translation.

- A is *asymptotically ρ-self-similar* about x if for any $r > 0$,

$$\lim_{n\to\infty} d_r(\rho^{n+1}\tau_{-x}A, \rho^n\tau_{-x}A) = 0$$

- A and B are *asymptotically similar* about 0 if for t complex,

$$\lim_{t\to\infty} d_r(tA, tB) = 0$$

Proposition 5.3 (Definition of scaling limit)**.** Let A be asympotically ρ-self-similar about a point x. Then there exists a unique ρ-self-similar set X such that $X \subset \bar{\mathbb{D}}_r$ and $(\rho^n\tau_{-x}A)_r \to X$ as $n \to \infty$ for all $r > 0$. We call this set the *scaling limit* or *limit model*.

Proof. Since $\{(\rho^n\tau_{-x}A)_r\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the Hausdorff metric, there is a unique limit X to this sequence. Then X satisfies $(\rho^n\tau_{-x}A)_r \to X$ as $n \to \infty$ for all $r > 0$. It follows that X is ρ-self-similar. □

Proposition 5.4. If A and B are asymptotically similar, and B is (asymptotically) ρ-self-similar, then A is also (asymptotically) ρ-self-similar.

Proof. Straightforward application of the triangle inequality. We have $\lim_{t\to\infty} d_r(tA, tB) = 0$ so that $\lim_{n\to\infty} d_r(\rho^n A, \rho^n B) = 0$ and as $(\rho^n B)_r \to (X)_r$ as $n \to \infty$ for some X, we also have $(\rho^n A)_r \to (X)_r$ as $n \to \infty$. □

5.2 Relation to holomorphic maps

This formalism provides another way to state the idea that a conformal map resembles locally translation followed by multiplication by a complex number.

Proposition 5.5. Let $\phi : U \to V$ be a holomorphic map that fixes the origin, with $\rho = \phi'(0) \neq 0$. Then for any compact set $A \subset U$, its image $\phi(A)$ is asymptotically similar to ρA about 0.

Proof. Pick any $r > 0$. If A does not accumulate at 0, the proof is trivial, so assume that 0 is indeed a limit point of A. Let $t \in \mathbb{C}$, then to each point $y \in t\rho A \cap \mathbb{D}_r$, there is a point x in A such that $y = t\rho x$, and since $y \in \mathbb{D}_r$ we have $|x| < \frac{r}{|\rho|}|t|^{-1}$, which can be made arbitrarily small by varying t. Let $\epsilon > 0$. Then

$$\begin{aligned} d\Big(y, t\phi(A)\Big) &= \min_{z\in\phi(A)} |t\rho x - tz| \\ &\leq |t\rho x - t\phi(x)| = |t||x| \left|\frac{\phi(x) - \rho x}{x}\right| \\ &\leq \frac{r}{|\rho|}\left|\frac{\phi(x) - \rho x}{x}\right| \end{aligned} \tag{16}$$

Since $\phi'(0) = \rho \neq 0$ we can choose $|t|$ large enough so that $\left|\frac{\phi(x)-\rho x}{x}\right| < \frac{|\rho|}{r}\epsilon$. Since the choice of $y \in t\rho A \cap \bar{\mathbb{D}}_r$ was arbitrary, we have:

$$\max_{y\in(t\rho A)_r} d\Big(y, t\phi(A)\Big) \underset{t\to\infty}{\longrightarrow} 0$$

Similarly, for any $y \in t\phi(A) \cap \bar{\mathbb{D}}_r$ there is a $x \in A$ such that $y = t\phi(x)$. Since both A and $\phi(A)$ are compact sets, and $\phi(z) = 0$ if and only if $z = 0$, the quantity $T = \max_{z\in A\setminus\{0\}} |z|\,|\phi(z)|^{-1}$ is defined and finite. Then $|x| \leq T|\phi(x)| = T|y/t| \leq Tr|t|^{-1}$. So

$$\begin{aligned} d\Big(y, t\rho A\Big) = \min_{z\in\rho A}|t\phi(x) - tz| &\leq |t\phi(x) - t\rho x| \\ &\leq Tr\left|\frac{\phi(x) - \rho x}{x}\right| \end{aligned} \tag{17}$$

which again can be made arbitrarily small by increasing $|t|$. So $\lim_{t\to\infty} d_r\Big(t\phi(A), t\rho A\Big) = 0$. □

Corollary 5.6. Let $\phi : U \to V$ be a conformal map and let A be a closed subset of U. Then for any a in A, the set A is asymptotically similar about a to its image $\phi(A)$ about $\phi(a)$, up to a rotation by $\lambda = \phi'(a)$. That is $\lim_{t\to\infty} d_r\Big(t\tau_{-a}A, t\lambda\tau_{-\phi(a)}\phi(A)\Big) = 0$.

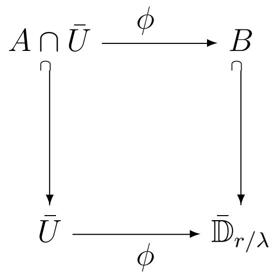
Proof. It suffices to notice that the map $\tau_{-\phi(a)} \circ \phi \circ \tau_{-a}$ is just as in proposition 5.5. □

6 Self-Similarity in the Julia set

Theorem 6.1. Let f be a rational map with repelling periodic point x of period p and multiplier λ. Let $A \ni x$ be a completely invariant closed set under f. Then A is asymptotically λ-self-similar at x, with limit model derived from the Kœnigs linearisation.

Proof. Let $g = f^p$ and $x = 0$ for simplicity. By Theorem 3.8, there exists a conformal biholomorphism ϕ from an open neighbourhood V of the origin to to another one V'

such that $\phi(0) = 0$, $\phi'(0) = 1$ and $\phi \circ g(z) = \lambda\phi(z)$ in V. We construct the limit set B in the following way. Since V' is open, there is $r > 0$ such that $\bar{\mathbb{D}}_r \Subset V'$. Define $U = \phi^{-1}(\mathbb{D}_{r/\lambda})$ and $B = \phi(A \cap \bar{U})$.



We know from Proposition 5.5 in the previous section that $A \cap \bar{U}$ and B are asymptotically similar about 0. We now prove that B is self-similar by showing that $(B)_{r/\lambda} = (\lambda B)_{r/\lambda}$

$$B \cap \mathbb{D}_{r/\lambda} = \phi(A \cap \bar{U}) \cap \phi(U) = \phi(A \cap \bar{U} \cap U) = \phi(A \cap U) \tag{18}$$

and, since $\phi \circ g(U) = \lambda\phi(U) = \mathbb{D}_r$, we have $U \subset g(U)$, and since A is invariant we have $\phi(A) = A$. Together these give:

$$\begin{aligned} \lambda B \cap \mathbb{D}_{r/\lambda} &= \lambda\phi(A \cap \bar{U}) \cap \phi(U) \\ &= \phi \circ g(A \cap \bar{U}) \cap \phi(U) \\ &= \phi(A \cap g(\bar{U}) \cap U) \\ &= \phi(A \cap g(\bar{U})) \cap \phi(U) \\ &= \phi(A \cap U) = B \cap \mathbb{D}_{r/\lambda} \end{aligned} \tag{19}$$

Thus proving that B is λ-self-similar about the origin. Thus by Proposition 5.4, A is asymptotically λ-self-similar about the origin. □

Corollary 6.2. Let z_0 be a repelling periodic point with multiplier λ of a rational map. The Julia set is then asymptotically λ-self-similar about z_0.

Theorem 6.3. Let f be a rational map with completely invariant closed set A and let x be an eventually periodic point of period p and multiplier λ such that $|\lambda| \neq 0, 1$. Then A is asymptotically λ-self-similar about x. If additionally, the orbit of x does not contain any critical point, then the limit model of A at x is the same of that of the cycle, up to multiplication by a constant.

Proof. Let $l \geq 1$ be minimal such that $f^p \circ f^l(x) = f^l(x)$ and let $\alpha = f^l(x)$. By the previous theorem, the set A is asymptotically λ-self-similar about α with limit model

$B = \phi(A \cap \bar{U})$ where ϕ is the Koenig linearisation such that $\phi'(\alpha) = 1$ and U is a certain neighbourhood of α in the domain of ϕ.

Suppose now that the orbit of x does not contain any critical point. Then $(f^l)'(x) \neq 0$ and so f^l maps a neighbourhood V of x conformally isomorphically to a neighbourhood of α. We may assume $U \subset f^l(V)$. Then $\phi \circ f^l$ maps $A \cap (f^l)^{-1}(\bar{U})$ isomorphically to B. Then by Proposition 5.5, A is λ-self-similar about x with limit model $(f^l)'(x)^{-1}B$. $\square$

We know that the iterated pre-images of a point z_0 in $J(f)$ are dense in $J(f)$. Then, provided at least one of the backward orbits

$$z_0 \xleftarrow{f} z_1 \xleftarrow{f} z_2 \xleftarrow{f} z_3 \xleftarrow{f} \cdots$$

has no critical points in it, the set of points at which J_c has the same limit model (up to rotation) is dense in J_c. In fact, there is only a finite set of points whose backwards orbits always hit a critical point. This explains why J_c looks the same everywhere.

7 The Filled Julia set varies continuously at Misiurewicz parameters

In this section we will prove the following:

Theorem 7.1. K_c varies continuously in the Hausdorff metric around Misiurewicz parameters.

This is a known result. See

To study the the behaviour of K_c as we vary $c \in \mathbb{C}$, we introduce the following sets:

$$\begin{aligned} K &= \{(c, z) \in \mathbb{C}^2 \mid z \in K_c\} \\ J &= \{(c, z) \in \mathbb{C}^2 \mid z \in J_c\} \end{aligned} \tag{20}$$

Incidentally this gives an alternative definition of the Mandelbrot set as the intersection of K with the graph of the identity function:

$$M = \{c \in \mathbb{C} \mid (c, c) \in K\} \tag{21}$$

Theorem 7.2. The set K is closed in $\mathbb{C}^2$.

Proof. Consider a convergent sequence $\{(c_n, z_n)\}_{n\in\mathbb{N}} \subset K$ and suppose for sake of contradiction that the limit (c_0, z_0) is not in K, so that $z_0 \notin K_{c_0}$. It is easy to check that for all $c \in \mathbb{C}$ there is a continuously varying escape radius $b(c) > 0$ such that

$K_c \subset \bar{\mathbb{D}}_{b(c)}$. Since the orbit of z_0 under iterations of f_{c_0} escapes to infinity, there is an integer i such that $|f^i_{c_0}(z_0)| > b(c_0)$. Since $|f^i_c(z)|$ and $b(c)$ vary continuously in c and z, all (c_n, z_n) sufficiently close to (c_0, z_0) satisfy $|f^i_{c_n}(z_n)| > b(c_n)$, which implies $z_n \notin K_{c_n}$, a contradiction. $\square$

The fact that K is closed is a necessary condition for the sets K_c to vary continuously with c. This is encapsulated in the following proposition, stated in a more general form.

Lemma 7.3. Let $\Delta \subset \mathbb{C}$ be closed and $X \subset \Delta \times \mathbb{C}$ such that for all $c \in \Delta$:

$$X_c = \{z \in \mathbb{C} \mid (c, z) \in X\} \neq \emptyset$$

Then X is closed if and only if for all $c_0 \in \Delta$, the set X_{c_0} is closed and, for all $r > 0$:

$$\lim_{c \to c_0} \delta\Big(\ (X_c)_r, (X_{c_0})_r\Big) = 0$$

Proof of lemma 7.3. Assume that X is closed. Fix c_0 in Δ, and we prove that X_{c_0} must be closed. To do so, notice that to any sequence $\{x_n\}_{n\in\mathbb{N}}$ in X_{c_0} converging to a point x_0, there corresponds a sequence $\{c_0, x_n\}$ in X that converges to the point $\{c_0, x_0\}$, which must be in X since this is closed. Therefore $x_0 \in X_{c_0}$, proving that X_{c_0} is closed. We prove the second statement by contradiction. Choose $r > 0$ and suppose for the sake of contradiction that there exists a sequence $\{c_n\}$ in Δ that converges to c_0 and $\epsilon > 0$ such that $\delta\Big(\ (X_{c_n})_r, (X_{c_0})_r\Big) > \epsilon$. This implies that for each n, there is x_n in $X_{c_n} \cap \bar{\mathbb{D}}_r$ such that $d\Big(x_n, (X_{c_0})_r\Big) > \epsilon$. Now, since the sequence $\{c_n, x_n\}$, at least for n large enough, lives in a compact subset of X, there is a subsequence $\{c_{n_k}, x_{n_k}\}$ that converges to a point (c_0, x_0) in X. By definition x is in $(X_{c_0})_r$. So $\{x_{n_k}\}$ converges to a point in $(X_{c_0})_r$ while $d\Big(x_{n_k}, (X_{c_0})_r\Big) > \epsilon$, which is a contradiction. These arguments can be repeated for any $c_0 \in \Delta$ and $r > 0$.

For the other direction, consider a sequence $\{c_n, x_n\}_{n\in\mathbb{N}}$ in X that converges to a point (c_0, x_0) in $\Delta \times \mathbb{C}$. To prove that X is closed it is sufficient to show that x_0 is in X_{c_0}. Let $d_0 = d(x_0, X_{c_0})$ and r large enough so that $X_{c_0} \cap \mathbb{D}_r \neq \emptyset$, x_0 is in $\mathbb{D}_r$ and $r > d_0$ so that $d(x_0, (X_{c_0})_r) = d_0$. Since $\{x_n\}$ converges to x_0, there is N such that for all n sufficiently large, x_n is so close to x_0 that $d(x_n, (X_{c_0})_r) = d(x_n, X_{c_0})$. We can then define a sequence $\{x_{n\perp}\}$ in X_{c_0} such that

$$x_{n\perp} \in \arg\min_{x \in X_{c_0}} |x_n - x|$$

but then

$$\begin{aligned} \min_{x \in X_{c_0}} |x_n - x| &= d\Big(x_n, X_{c_0}\Big) \\ &= d(x_n, (X_{c_0})_r) \\ &\leq \delta\Big(X_{c_n}, (X_{c_0})_r\Big) \underset{n\to\infty}{\longrightarrow} 0 \end{aligned} \tag{22}$$

So then $|x_n - x_{n\perp}| \to 0$ and therefore $\lim_{n\to\infty} x_{n\perp} = \lim_{n\to\infty} x_n = x_0$. Since X_{c_0} is closed by assumption and x_0 is the limit of a sequence in X_{c_0}, x_0 is in X_{c_0}. $\square$

Corollary 7.4. For all $c \in \mathbb{C}$, and all $r > 0$, $\lim_{c'\to c} \delta\Big((K_{c'})_r, (K_c)_r\Big) = 0$.

We have learned something about the variation of the filled Julia set. The above corollary indicates that, even if there are discontinuities, they are of a kind, which can be called "implosive". To make it explicit we consider simple example. Let

$$X = (\bar{\mathbb{D}}_2 \times \bar{\mathbb{D}}) \cup (\bar{\mathbb{D}} \times \bar{\mathbb{D}}_2)$$

so that

$$X_c = \begin{cases} \bar{\mathbb{D}}_2 & \text{for } c \in \bar{\mathbb{D}} \\ \bar{\mathbb{D}} & \text{for } c \in \bar{\mathbb{D}}_2 \backslash \bar{\mathbb{D}} \\ \emptyset & \text{for } c \notin \bar{\mathbb{D}}_2 \end{cases}$$

As c varies from 0 to 3 on the real line, X_c suffers two "implosions", first from $\bar{\mathbb{D}}_2$ to $\bar{\mathbb{D}}$ and then to $\emptyset$. These are hardly continuous changes. However since X is closed by virtue of being the union of two closed sets, by Lemma 7.3 we must have $\delta(\bar{\mathbb{D}}, \bar{\mathbb{D}}_2) = 0$ and this indeed the case. As discussed in Section 5, the only requirement for $K_{c'}$ as c' approaches c is that it must be a subset of $N_\epsilon(K_c)$ for smaller and smaller values of ϵ. This allows for events called parabolic implosions (Figure 4) see [21].

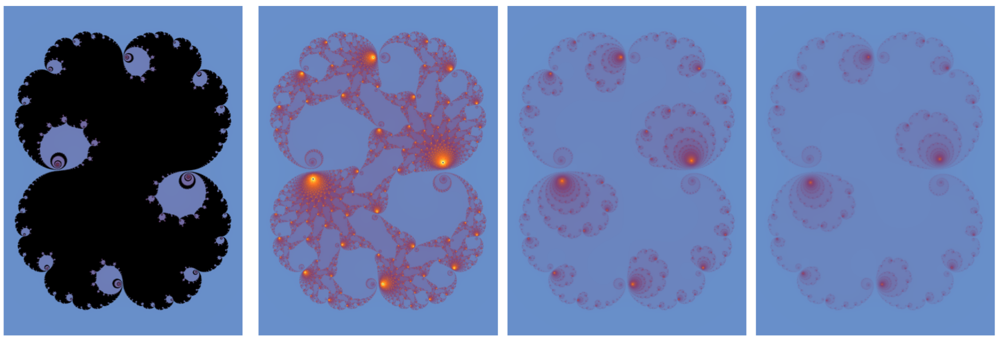


Figure 4: *An example of parabolic implosion*: Representations of K_c as c varies from $0.252 + 0.0002i$ to $0.258 + 0.0008i$ in equal steps.

To have continuity in the Hausdorff distance, there needs to be more constraints. For this we introduce the concept of a continuous section, which will provide some more control on the variation of K_c.

Definition 13. Given a set $X \subset \mathbb{C} \times \mathbb{C}$. A *continuous local section* of X at $(c_0, z_0) \in X$ is a continuous map h_{z_0} from a neighbourhood $U \ni c_0$ to $\mathbb{C}$ such that $h_{z_0}(c_0) = z_0$ and for all $c \in U$, $h_{z_0}(c) \in X_c$.

Proposition 7.5. Let $\Delta \subset \mathbb{C}$ be closed and $X \subset \Delta \times \mathbb{C}$ such that for all $c \in \Delta$:

$$X_c = \{z \in \mathbb{C} \mid (c, z) \in X\} \neq \emptyset$$

Suppose additionally that for some c_0 in Δ, there a is set A, dense in X_{c_0} such that, for all a in A, there is a continuous local section h_a of X at (c_0, a). Then X_c varies continuously around c_0 in the sense that: $\lim_{c \to c_0} d_r\Big(X_c, X_{c_0}\Big) = 0$ for all $r > 0$.

Proof. For simplicity, assume that X is translated so that $c_0 = 0$. Choose $r > 0$. By Lemma 7.3 we know $\lim_{c\to 0} \delta\Big(\ (X_c)_r, (X_0)_r\Big) = 0$. We need to prove $\lim_{c\to 0} \delta\Big(\ (X_0)_r, (X_c)_r\Big) = 0$.

Set $\epsilon > 0$; the set of disks of radius $\epsilon/2$ centred at the elements of A provide a cover of X_0, and hence of $X_0 \cap \bar{\mathbb{D}}_r$. The latter being compact, there exists a finite set of points $a_1, a_2, \cdots, a_m \in A$ such that

$$X_0 \cap \bar{\mathbb{D}}_r \subset \bigcup_{i=1}^{m} \mathbb{D}_{\epsilon/2}(a_i)$$

By hypothesis, there is a continuous local section at each of the a_i. By the continuity, for each $i = 1, 2, \cdots, m$ there is $\eta_i > 0$ such that $|h_{a_i}(c) - a_i| < \epsilon/2$ as soon as $|c| < \eta_i$. Set $\eta = \min \eta_i$. Now pick any $c \in \mathbb{D}_\eta$. For every point $x_0 \in X_0 \cap \bar{\mathbb{D}}_r$ we have $d\,(x_0, (X_c)_r) \leq d\,(x_0, X_c)$ and

$$\begin{aligned} d\,(x_0, X_c) &= \min_{x \in X_c} |x - y| \\ &\leq \min_i |x - h_{a_i}(c)| \\ &\leq \min_i |x - a_i| + |h_{a_i}(c) - a_i| \\ &\leq \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

and thus $\delta\Big(\ (X_0)_r, (X_c)_r\Big) = \delta\Big(X_0 \cap \bar{\mathbb{D}}_r, (X_c)_r\Big) < \epsilon$. □

Proof of Theorem 7.1. We know that the set of repelling periodic orbits is dense in J_c, and the implicit function theorem makes it possible to create continuous local sections through each of these. We have also proven that when c_0 is Misiurewicz, $J_{c_0} = K_{c_0}$. Thus Proposition 7.5 applies and

$$\lim_{c \to c_0} d_r(K_c, K_{c_0})$$

□

Note that this proof relies crucially on the fact that c_0 is Misiurewicz. Generally however, f_c might allow for parabolic points, which for small variations of c will vanish.

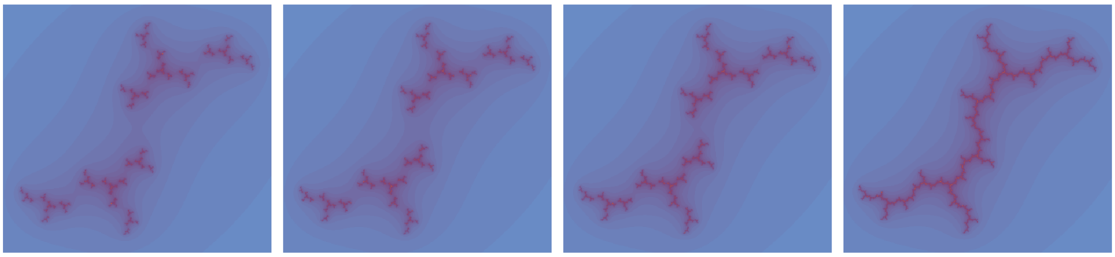


Figure 5: *Continuous variation of* K_c: Representations of K_c for c approaching the Misiurewicz parameter $c_0 = -i$ from the left.

8 Self-Similarity in the Mandelbrot set

The following section will prove the main result in Tan Lei's paper [3] concerning the self-similairty of the Mandelbrot set at the Misiurewicz parameters. Tan Lei goes about the proof in two steps. In Section 4 of the paper, she proves a general result for continuous mappings from $\mathbb{C} \to \mathbb{C}^k$, and then in Section 5 she verifies that the a map $u : \mathbb{C} \to \mathbb{C}$, to be introduced below, connecting the dynamical plane to the parameter plane satisfies the conditions for the proposition of Section 4. In work, we prove the result directly for the Mandelbrot set, without deriving the general case. Taking this direct path perhaps makes the logic easier to follow. Some details of the proof are expanded upon.

8.1 Setup

Let $c_0 \in \mathbb{C}$ be a Misiurewicz parameter with l, p minimal such that

$$f_{c_0}^p \circ f_{c_0}^l(c_0) = f_{c_0}^l(c_0)$$

We recall from Section 4.1 then that $\alpha_0 \equiv f_{c_0}^l(c_0)$ is a repelling periodic point with multiplier $\rho_0 \equiv (f_{c_0}^p)'(\alpha_0)$ and that $(f_{c_0}^l)'(c_0) \neq 0$. K_{c_0} coincides with J_{c_0} and is asymptotically ρ_0-self-similar about both α_0 and c_0 from Section 6.2.

We can apply the implicit function theorem to obtain a holomorphic function $\alpha(c)$ defined in a neighbourhood Δ of c_0 such that $\alpha(c_0) = \alpha_0$, and $f_c^p(\alpha(c)) = \alpha(c)$. The neighbourhood can be taken small enough so that $\alpha(c)$ is a repelling p-periodic point under f_c for all $c \in \Delta$. Write $\rho(c) = (f_c^p)'(\alpha(c))$ for the multiplier of these orbits.

Then for all parameters $c \in \Delta$, K_c will be asymptotically $\rho(c)$-self-similar about both $\alpha(c)$ Let X_c be the scaling limit of K_c at $\alpha(c)$, and recall by Theorem 6.1 that X_c is obtained by using the Koenig linearisation $\phi_{\alpha(c)}$.

The key to connecting the dynamical and parameter planes is the function

$$u(c) = \phi_{\alpha(c)} \circ f_c^l(c) \tag{23}$$

as will become evident later. The fact that this is a holomorphic function is non-trivial, as $\phi_{\alpha(c)}$ Kœnigs linearisations for different maps. Tan Lei states without proof $(c, z) \mapsto \phi_{\alpha(c)}(z)$ depends holomorphically in c in a neighbourhood of (c_0, α_0) (Lemma 5.2 of the paper). Here we provide proof for the following two statements that will be used to prove the asymptotic self-similarity of the Mandelbrot set at c_0.

Proposition 8.1. The map $\phi : (c, z) \mapsto \Big(c, \phi_{\alpha(c)}(z)\Big)$, where $\phi_{\alpha(c)}$ denotes the Koenig linearisation of f_c at $\alpha(c)$, is holomorphic in a neighbourhood of (c_0, α_0).

Corollary 8.2. The map $\Phi : (c, z) \mapsto \Big(c, \phi_{\alpha(c)} \circ f_c^l(z)\Big)$ is holomorphic in a neighbourhood U of (c_0, c_0).

Corollary 8.2 in turn implies that u is holomorphic in a neighbourhood of c_0.

Lemma 8.3. Let $f : \mathbb{D}_r(a) \to \mathbb{C}$ be holomorphic. If $f'(a) \neq 0$ for all $z \in \mathbb{D}_r(a)$ we have $|f'(z) - f'(a)| < |f'(a)|$ then f is invertible.

Proof. We may assume without loss of generality that $r = 1$ and $a = 0$. Notice that the inequality implies $f'(z) \neq 0$. Fix $\omega \in \mathbb{C}$, and consider the function

$$g(z) = z + \frac{\omega - f(z)}{f'(0)}$$

And notice that

$$|g'(z)| = \left|1 - \frac{f'(z)}{f'(0)}\right| = \left|\frac{f'(0) - f'(z)}{f'(0)}\right| < 1$$

Additionally, notice that $g(z) = z \iff f(z) = \omega$. Thus, any solution for $f(z) = \omega$ for $z \in \mathbb{D}$ will be an attracting point of the dynamics of g. Let x be such a solution; we prove there cannot be another one. Then we can use the conformal isomorphism of the disk:

$$\phi_x : z \longmapsto \frac{z - x}{1 - \bar{x}z}$$

to conjugate the dynamics of g into those of $\tilde{g} = \phi_x \circ g \circ \phi_x^{-1}$. Now the origin is an attractive fixed point of $\tilde{g}$. Take any $z \in \mathbb{D}$, then we integrate along the straight line connecting it with the origin to learn that the map is strictly contracting,

$$|g(z)| = \left|\int_0^z g'(u)\mathrm{d}u\right| \leq \int_0^z |f'(z)|\, |\mathrm{d}z| < \int_0^z |\mathrm{d}z| = |z|$$

so that no other point can fixed. □

Lemma 8.4. Let $g : \mathbb{C}^2 \to \mathbb{R}$ be a continuous function and let α be a holomorphic function in one complex variable. Suppose $g(z, \alpha(c_0)) < 0$ holds for all z in a neighbourhood of c_0 for some c_0. Then there exists a disk Δ centred at c_0 and a disk D centred at $\alpha(c_0)$ such that $\alpha(\Delta) \Subset D$ and $g(z, \alpha(c)) < 0$ for all $z \in D$ and $c \in \Delta$.

Proof. Assume without loss of generality that $\alpha(c_0) = 0$. There is $r > 0$ such that for all $z \in \mathbb{D}_{5r}$, $g(z, 0) < 0$. Since α is holomorphic, there is $\delta_1 > 0$ such that, for $\Delta = \mathbb{D}_{\delta_1}(c_0)$, $\alpha(\Delta) \Subset \mathbb{D}_r$, so that for all $c \in \Delta$, we have

$$\alpha\Big(\Delta\Big) \Subset \mathbb{D}_r \Subset \mathbb{D}_{2r}(\alpha(c)) \Subset \mathbb{D}_{4r}$$

We define the set

$$V = \bigcap_{c \in \Delta} \mathbb{D}_{2r}\Big(\alpha(c)\Big)$$

this set is non-empty, as it contains $\alpha(\Delta)$ and is compactly contained in $\mathbb{D}_{5r}$. We show that we can select a neighbourhood of c_0 so that $g(z, \alpha(c)) < 0$ for all $z \in V$.

For each $z \in V$, thanks to the continuity of g and the analyticity of α, there is $\delta(z) > 0$ such that $g(z, \alpha(c)) < 0$ for all $c \in \mathbb{D}_{\delta(z)}(c_0)$. Since V is compact, $\delta(z)$ achieves a minimum $\delta_2 > 0$. Set $\Delta' = \mathbb{D}_{\min\{\delta_1, \delta_2\}}(c_0)$. It is easy to check $\mathbb{D}_r \subset V$.We then have the following properties:

$$\alpha(\Delta') \in \mathbb{D}_r \quad \text{and} \quad g(z, \alpha(c)) < 0$$

for all $z \in \mathbb{D}_r$ and $c \in \Delta'$. □

Proof of Proposition 8.1. We first prove that we can restrict Δ so that there is neighbourhood of V of α_0 such that $\alpha(\Delta) \subset V$ and $f_c^p|_V$ has a well defined inverse. For all z in a certain neighbourhood of α_0,

$$|(f^p)'_{c_0}(z) - (f^p)'_{c_0}(\alpha_0)| < |(f^p)'_{c_0}(\alpha_0)|$$

We can then apply Lemmas 8.4 and 8.3 to obtain two disks Δ and D centred at c_0 and α_0 respectively, by $f_c^p|_D$ is invertible for all $c \in \Delta$.

Denote by g_c the inverse of $f_c^p|_D$ and $\rho(c) = (g_c)'(\alpha(c)) = \lambda(c)^{-1}$. Consider the functions

$$\phi_{c,n} : z \longmapsto \frac{g_c^n(z) - \alpha(c)}{\rho(c)^n} \tag{24}$$

For each n and each c, they obey: $\phi_{c,n} \circ g_c(z) = \rho(c)\phi_{c,n+1}(z)$ so that, wherever they converge, they converge to the Koenig linearisation $\phi_{\alpha(c)}$ of g_c at $\alpha(c)$. We prove that there is a neighbourhood of (c_0, α_0) where they converge uniformly.

Let $\rho_1 = \inf_{c \in \Delta} |\rho(c)|$ and $\rho_2 = \sup_{c \in \Delta} |\rho(c)|$. Since $|\rho_0|^2 < |\rho_0|$ we can assume that Δ is small enough so that $\rho_2^2 < \rho_1$ and so we can choose $a > 0$ so that $0 < a^2 < \rho_1 < \rho_2 < a < 1$. We can then take $r_1' > 0$ so that for all $z \in \mathbb{D}_{r_1'}(\alpha_0)$ we have $|g_{c_0}(z) - \alpha_0| < a|z - \alpha_0|$. By applying Lemma 8.4 again, we can find $\delta_1, r_1 > 0$, such that, for all $c \in \mathbb{D}_{\delta_1}(c_0)$ and $z \in \mathbb{D}_{r_1}(\alpha_0)$

$$|g_c(z) - \alpha(c)| < a|z - \alpha(c)|$$

and by induction,

$$|g_c^n(z) - \alpha(c)| < a^n |z - \alpha(c)|$$

Additionally, there is $r_2' > 0$ and $A > 0$ such that for all $z \in \mathbb{D}_{r_2'}(\alpha_0)$

$$|g_{c_0}(z) - \alpha_- \rho_0(z - \alpha_0)| < A|z - \alpha_0|^2$$

Applying Lemma 8.4 once more, we obtain $\delta_2, r_2 > 0$, such that, for all $c \in \mathbb{D}_{\delta_2}(c_0)$ and $z \in \mathbb{D}_{r_2}(\alpha_0)$

$$|g_c(z) - \alpha(c) - \rho(c)(z - \alpha(c))| < A|z - \alpha(c)|^2$$

We now set $\delta = \min\{\delta_1, \delta_2\}$ and $r = \min\{r_1, r_2\}$ so that, for all $c \in \mathbb{D}_\delta(c_0)$ and $z \in \mathbb{D}_r(\alpha_0)$ the two properties above hold, and we can deduce that

$$|g_c^{n+1}(z) - \alpha(c) - \rho(c)(g_c^n(z) - \alpha(c))| < A|g_c^n(z) - \alpha(c)|^2 < Aa^{2n}r'$$

Finally, we have

$$\begin{aligned} |\phi_{c,n+1}(z) - \phi_{c,n}(z)| &= \left| \frac{g_c^{n+1}(z) - \alpha(c)}{\rho(c)^{n+1}} - \frac{g_c^n(z) - \alpha(c)}{\rho(c)^n} \right| \\ &= \rho(c)^{-(n+1)} ||g_c^{n+1}(z) - \alpha(c) - \rho(c)(g_c^n(z) - \alpha(c))| \qquad (25) \\ &< \rho_1^{-(n+1)} Aa^{2n}r' = \frac{Ar'}{\rho_1} \left(\frac{a^2}{\rho_1} \right)^n \underset{n\to\infty}{\longrightarrow} 0 \quad \text{since } \rho_1 > a^2 \end{aligned}$$

So that each sequence $\phi_{c,n}$ converges to a holomorphic function ϕ_c as $n \to \infty$, uniformly in c, which means that $\phi_c(z)$ meaning that holomorphically on c. ☐

Proof of Corollary 8.2. This follows immediately from the Lemma 8.4 and the fact that $(c, z) \mapsto (c, f_c^l(z))$ is holomorphic. ☐

Another key property, known as *transversality* is that $u'(c_0) \neq 0$. This is a non trivial result with a number of proofs, see below.

By the previous lemmas, we can assume that there is $\delta > 0$ and $V \ni c_0$ such that, letting $\Delta = \bar{\mathbb{D}}_\delta(c_0)$ for all $c \in \Delta$, the function $\phi_{\alpha(c)} \circ f_c^l|_V$ has a holomorphic inverse. We may assume that $\Delta \Subset V$. Restrict Δ further if necessary so that we can also choose $r > 0$ such that for all $c \in \Delta$:

$$\Phi(c, c) = \Big(c, u(c)\Big) \in \Delta \times \bar{\mathbb{D}}_r \subset \Phi(\Delta \times V)$$

Define

$$\Omega = \Phi^{-1}(\Delta \times \bar{\mathbb{D}}_r) \qquad (26)$$

which is closed because it is the preimage of a closed set. By Lemma 7.3, the sets

$$\Omega_c = \{z \in \mathbb{C} \mid (c, z) \in \Omega\} \qquad (27)$$

are also closed. By definition $c \in \Omega_c$. Essentially ω is the set where the $K\oe nigs$ linearisations

We also collect the models

$$X_c = \phi_c \circ f_c^l(K_c \cap \Omega_c) \tag{28}$$

into the set

$$X = \Phi(K \cap \Omega) = \{(c, x) \in \Omega \mid x \in X_c\} \tag{29}$$

Since K and Ω are closed sets and Φ is analytical, X is also a closed set. Notice that the local continuous sections of K at repelling periodic points of K_{c_0} are mapped by Φ to local continuous sections of X on a dense subset of X_{c_0}. We then have by Proposition 7.5:

$$\lim_{c \to c_0} X_c = X_{c_0} \tag{30}$$

Notice that, for all $c \in \Delta$, $u(c)$ is in X_c if and only if c is in K_c, which itself is equivalent to saying that c is in the Mandelbrot set M. So we can write:

$$M \cap \Delta = \{c \in \Delta \mid u(c) \in X_c\} = u(\Delta) \cap X \tag{31}$$

And we can finally state and prove the main theorem.

Theorem 8.5 (Tan Lei)**.** The Mandelbrot set is asymptotically ρ_0-self-similar about c_0. In particular, there exists a radius $s > 0$ such that if we write $M_n = \rho_0^n u'(c_0)\tau_{-c_0}(M \cap \Delta)$ for all $n \in \mathbb{N}$, then

$$\lim_{n \to \infty} d_s(M_n, X_{c_0}) = 0 \tag{32}$$

Proof. We list a number of properties that are possible for all $c \in \Delta$ because Δ is compact:

- There is a radius $s > 0$ such that $\Big(\rho(c)X_c\Big)_s = (X_c)_s$ for all $c \in \Delta$.
- There are constants $C_1, C_2, C_3 > 0$ such that:

$$\begin{aligned} |\rho(c) - \rho(c_0)| &\leq C_1|c - c_0| \\ |u(c)| &\leq C_2|c - c_0| \\ |u(c) - u'(c_0)(c - c_0)| &\leq C_3|c - c_0|^2 \end{aligned} \tag{33}$$

- Since $|\rho_0||\rho_0|^{-2} < 1$, we can assume Δ is small enough so that $\mu_1\mu_2^2 < 1$, where

$$\begin{aligned} \mu_1 &= \max_{c \in \Delta} |\rho(c)| \\ \mu_2 &= \max_{c \in \Delta} |\rho(c)|^{-1} \end{aligned} \tag{34}$$

Define for all $n \in \mathbb{N}$:

$$M_n = \rho_0^n u'(c_0)\tau_{-c_0} M_\Delta \tag{35}$$

Step 1. We first prove that $\delta\Big((M_n)_s, (X_{c_0})_s\Big) \longrightarrow 0$ as $n \to \infty$.

A point y is in $M_n \cap \bar{\mathbb{D}}_s$ if and only if there is a parameter $c \in M \cap \Delta$ such that $y = \rho_0^n u'(c_0)(c - c_0)$. This parameter then satisfies

$$|c - c_0| = |\rho_0^n u'(c_0)|^{-1} |y| \leq \left|\frac{s}{u'(c_0)}\right| |\rho_0|^{-n} \leq C|\rho_0|^{-n} \leq C\mu_2^n$$

To estimate the distance from a point y in $M_n \cap \bar{\mathbb{D}}_s$ to $(X_{c_0})_s$ we use the triangle inequality

$$d\Big(y, (X_{c_0})_s\Big) \leq |y - \rho(c)^n u(c)| + d\Big(\rho(c)^n u(c), (X_{c_0})_s\Big) \tag{36}$$

Let's look at each summand of (36) in turn. We split the first one in two terms, again using the triangle inequality:

$$\begin{aligned} |y - \rho(c)^n u(c)| &= |\rho_0^n u'(c_0)(c - c_0) - \rho_0^n u(c)| + |\rho_0^n u(c) - \rho(c)^n u(c)| \\ &\leq |\rho_0|^n |u'(c_0)(c - c_0) - u(c)| + |\rho_0^n - \rho(c)^n||u(c)| \end{aligned} \tag{37}$$

For the first term, we can directly apply one of the continuity relation above and the bound on $|c - c_0|$:

$$|\rho_0|^n |u'(c_0)(c - c_0) - u(c)| \leq |\rho_0|^n C_3 |c - c_0|^2 \leq CC_3 |\rho_0|^{-n} \underset{n\to\infty}{\longrightarrow} 0$$

For the second term, we make use of a telescoping sum, and then apply two of the continuity relations and the bound on $|c - c_0|$:

$$\begin{aligned} |\rho_0^n - \rho(c)^n| &= \left|\sum_{i=0}^{n-1} \rho_0^{n-i}\rho(c)^i - \rho_0^{n-1-i}\rho(c)^{i+1}\right| \\ &= \left|(\rho(c) - \rho_0) \sum_{i=0}^{n-1} \rho_0^{n-1-i}\rho(c)^i\right| \\ &\leq |\rho(c) - \rho_0| \sum_{i=0}^{n-1} |\rho_0|^{n-1-i}|\rho(c)|^i \leq |\rho(c) - \rho_0| n\mu_1^{n-1} \\ |\rho_0^n - \rho(c)^n| &\leq C_1|c - c_0|\, n\mu_1^{n-1} \end{aligned}$$

$$\begin{aligned} \Rightarrow |\rho_0^n - \rho(c)^n|\, |u(c)| &\leq C_1 n\mu_1^{n-1}|c - c_0| C_2 |c - c_0| = C_1C_2|c - c_0|^2 n\mu_1^{n-1} \\ &\leq C_1C_2C^2\mu_2^{2n}\, n\mu_1^{n-1} \\ &\leq C_1C_2C^2\, n(\mu_1\mu_2^2)^n \underset{n\to\infty}{\longrightarrow} 0 \end{aligned}$$

Thus the second term also tends to 0, uniformly in y.

Fix $\epsilon > 0$. We have just proven that there is N_1 such that for all $n > N_1$, we have

$$|y - \rho(c)^n u(c)| < \epsilon/2$$

and, as $y \in \bar{\mathbb{D}}_r$, we have

$$|\rho(c)^n u(c)| < s + \epsilon/2$$

Since $u(c)$ is in X_c, which is $\rho(c)$-self similar in the window of radius s, $\rho(c)^n u(c)$ is in $(X_c)_s$, provided that $|\rho(c)^n u(c)| \leq s$. Additionally, since $(X_c)_s$ contains the circle of radius s, we conclude that, in any case:

$$d\Big(\rho(c)^n u(c), (X_c)_s\Big) < \epsilon/2$$

Moreover,

$$d\Big(\rho(c)^n u(c), (X_{c_0})_s\Big) \leq d\Big(\rho(c)^n u(c), (X_c)_s) + d_s(X_c, X_{c_0}\Big)$$

and since $d_s(X_c, X_{c_0}\Big) \to 0$ as $c \to c_0$, and $c \to c_0$ as $n \to \infty$, we can choose N_2 so that for all $n > N_2$ we have $d_s(X_c, X_{c_0}\Big) < \epsilon/2$, so that

$$d\Big(\rho(c)^n u(c), (X_{c_0})_s\Big) < \epsilon$$

This completes the proof that $\delta\Big(\ M_n, (X_{c_0})_s\Big)$ tends to 0 with n.

Step 2. We now prove that $\delta\Big(\ (X_{c_0})_s, (M_n)_s\Big) \longrightarrow 0$ when $n \to \infty$.

Let $A \in X_{c_0}$ denote the dense subset of the images under the map $z \mapsto \phi_{\alpha(c)} \circ f_c^l(z)$ of the repelling periodic (under f_c) points in the neighbourhood in $K_c \cap \Omega_c$. We can show that for each $a \in A \cap \bar{\mathbb{D}}_s$ $d(a, (M_n)_s) \to 0$. Fix a in A, then there is a continuous section $h_a : \mathbb{D}_d(c_0) \to X$ such that

$$h_a(c_0) = a \quad \text{and } h_a(c) \in X_c \cap \bar{\mathbb{D}}_s$$

Since $u(c_0) = 0$ and $u'(c) \neq 0$, and $h_a\Big(\mathbb{D}_d(c_0)\Big)$ is in the disk $\mathbb{D}_s$, Lemma 8.6 below guarantees that the equation

$$u(c) - \rho(c)^{-n} h_a(c) = 0 \tag{38}$$

has a solution in $\mathbb{D}_d$, at least for n large enough. We can then build a sequence $\{c_n\}$ such that

$$\rho(c_n)^n u(c_n) = h_a(c_n) \tag{39}$$

so that $\rho(c_n)^n u(c_n)$ must be in $X_{c_n} \cap \bar{\mathbb{D}}_s$.

Since this set is $\rho(c_n)$-self-similar, $u(c_n)$ is also in $(X_{c_n})_s$ for n and hence c_n is in $M \cap \Delta$ and $\rho_0^n u'(c_0)(c_n - c_0) \in M_n$. Then

$$\begin{aligned} d\Big(a, (M_n)_s\Big) \leq d(a, M_n) &\leq |a - \rho_0^n u'(c_0)(c_n - c_0)| \\ &\leq |a - h_a(c_n)| + |\rho(c_n)^n u(c_n) - \rho_0^n u'(c_0)(c_n - c_0)| \end{aligned} \tag{40}$$

We prove that c_n converges to c_0 geometrically. Indeed, one of the continuity conditions gives

$$|u'(c_0)(c_n - c_0)| \leq |u(c_n)| + C_3 |c_n - c_0|^2$$

so that if d is chosen small enough that $dC_3 \leq |u'(c_0)|$. This can always be done, and independently of a. Then, since $|c_n - c_0| < d$, we have

$$\begin{aligned} u'(c_0)(c_n - c_0)| &\leq |u(c_n)| + C_3 d |c_n - c_0| \\ \Rightarrow |c_n - c_0| &\leq \frac{1}{|u'(c_0)| - C_3 d} |u(c_n)| = C' |u(c_n)| \end{aligned} \tag{41}$$

Now, since $\rho(c_n)^n u(c_n)$ is in $X_{c_n} \cap \bar{\mathbb{D}}_s$, we have $|u(c_n)| \leq |\rho(c_n)|^{-n} s \leq \mu_2^n s$ and thus

$$|c_n - c_0| \leq C' \mu_2^n s \underset{n \to \infty}{\to} 0 \tag{42}$$

This last equation lets us prove the convergence of (40). Indeed, the first summand tends to 0 because of the continuity of h_a. The second summand is bounded by

$$|\rho(c_n)^n - \rho(c_0)^n||u(c_n)| + |\rho_0^n||u(c_n) - u'(c_0)(c_n - c_0)|$$

which can be treated just like (37) in Step 1.

Now we can prove that $\delta\Big(\ (X_{c_0})_s, (M_n)_s\Big)$ goes to 0. Given $\epsilon > 0$ we can pick a finite set $a_1, a_2, \cdots, a_m \in A$ so that

$$X_{c_0} \cap \bar{\mathbb{D}}_s \subset \bigcup_{i=1}^{m} \mathbb{D}_{\epsilon/2}(a_i)$$

for each of the a_i there is $d_i \leq |u'(c_0)|/C_3$ and we can choose $d = \min_i d_i$. Then for all $c \in \mathbb{D}_d$, and all a_i,

$$d\Big(a_i, (M_n)_s\Big) \underset{n \to \infty}{\to} 0$$

uniformly, so that there is N such that for all $n > N$ and all i, $d\Big(a_i, (M_n)_s\Big) \leq \epsilon/2$. Then for any $x_0 \in X_{c_0} \cap \bar{\mathbb{D}}_s$:

$$\begin{aligned} d\Big(\ x_0, (M_n)_s\Big) &\leq \min_i |x - a_i| + d\Big(\ a_i, (M_n)_s\Big) \\ &\leq \epsilon/2 + \epsilon/2 = \epsilon \end{aligned} \tag{43}$$

□

A remark on the proof. A loose description of the proof can be as follows. The function $z \mapsto \phi_{\alpha(c)} \circ f_c^l(z)$ is a function takes a neighbourhood of the dynamical plane to the Kœnigs coordinates for the parameter c. Then $c \mapsto u(c) = \phi_{\alpha(c)} \circ f_c^l(c)$ can be thought as taking each point c of the parameter space, injecting it in the dynamical plane for f_c and then sending that to the associated Kœnigs plane. The $c \in M$ are exactly those that land on the the associated X_c. We take these and project them from their dynamical plane to that of c_0. The fact that $X_c \to X_{c_0}$ ensures us that the projection $u(c)$ is not never too far from X_{c_0}. Transitivity ensures that the "projection" $u(M)$ is spread out in the dynamical plane of c_0.

Lemma 8.6. Let $u : V \to \mathbb{C}$ be an holomorphic function that fixes the origin, such that $u'(0) \neq 0$, then there is a constant $\eta > 0$ such that, for all holomorphic functions $v : \mathbb{D} \to \mathbb{D}_\eta$, the equation $u(z) + v(z) = 0$ always has a solution.

Proof. There is a disk $\mathbb{D}_\epsilon$ that is mapped univalently by u, and since 0 will be in the image, there is $\eta > 0$ such that $\mathbb{D}_{4\eta} \Subset u(\mathbb{D}_\epsilon)$. Then $\gamma = u^{-1}(\partial \mathbb{D}_{3\eta})$ is a simple closed curve. Now consider any holomorphic $v : \mathbb{D} \to \mathbb{D}_\eta$. The image of γ under $u + tv$, for any $t \in [0, 1]$ must be a closed curve in the annulus $\mathbb{D}_{4\eta} \backslash \bar{\mathbb{D}}_\eta$. As a consequence, the component of $V \backslash \gamma$ that contains 0 gets mapped by $u + v$ to a set that contains $\mathbb{D}_\eta$ as a subset, and hence 0. □

The proof crucially relies on this lemma:

Lemma 8.7 (Transversality). $u'(c_0) \neq 0$

Proof. We define two functions:

$$\begin{aligned} \beta : \Delta &\longrightarrow \mathbb{C} \\ c &\longmapsto f_c^l(c) \\ \\ w : \Delta &\longrightarrow \mathbb{C} \\ c &\longmapsto f_c^p\big(\beta(c)\big) - \beta(c) \end{aligned} \tag{44}$$

The first tracks the point where c lands in a neighbourhood of the periodic $\alpha(c)$ after l iterations and the second can be seen as a measure of how far off $f_c^l(c)$ is from being periodic. Indeed $\beta(c_0) = \alpha_0$ and $w(c_0) = 0$. We will first prove

$$u'(c_0) = \beta'(c_0) - \alpha'(c_0) = (\rho_0 - 1)^{-1} w'(c_0)$$

To do this, introduce the function $F(c, z) = \phi_{\alpha(c)}(z)$. Then $u(c) = \phi_{\alpha(c)} \circ f_c^l(c) = F\big(c, \beta(c)\big)$ and thus

$$u'(c_0) = \partial_c F(c_0, \alpha_0) + \beta'(c_0) \partial_z F(c_0, \alpha_0)$$

We have

$$\partial_z F(c_0, \alpha_0) = (\phi_{\alpha_0})'(\alpha_0) = 1$$

by definition. Then notice that $F(c, \alpha(c)) = 0$ identically, so that

$$\frac{F(c,\alpha_0) - F(c_0,\alpha_0)}{c - c_0} = \frac{F(c,\alpha_0) - F(c,\alpha(c))}{c - c_0} = -\frac{F(c,\alpha_0) - F(c,\alpha(c))}{\alpha(c) - \alpha_0}\frac{\alpha(c) - \alpha_0}{c - c_0}$$

$$\Rightarrow \partial_c F(c_0,\alpha_0) = \lim_{c\to c_0} \frac{F(c,\alpha_0) - F(c_0,\alpha_0)}{c - c_0} = -\partial_z F(c_0,\alpha_0)\alpha'(c_0) = -\alpha'(c_0)$$

so that indeed $u'(c_0) = \beta'(c_0) - \alpha'(c_0)$.

Now, since $f_c^p \circ \alpha(c) = \alpha(c)$ and $w(c_0) = 0$,

$$\begin{aligned}\frac{w(c) - w(c_0)}{c - c_0} &= \frac{w(c)}{c - c_0}\\ &= \frac{f_c^p \circ \beta(c) - f_c^p \circ \alpha(c)}{c - c_0} + \frac{\alpha(c) - \beta(c)}{c - c_0}\\ &= \left(\frac{f_c^p \circ \beta(c) - f_c^p \circ \alpha(c)}{\beta(c) - \alpha(c)} - 1\right)\frac{\beta(c) - \alpha(c)}{c - c_0}\end{aligned}$$

$$\Rightarrow w'(c_0) = \Big((f_{c_0}^p)'(c_0) - 1\Big)\left(\beta'(c_0) - \alpha'(c_0)\right) = (\rho_0 - 1)\left(\beta'(c_0) - \alpha'(c_0)\right)$$

The fact that $w'(c_0) \neq 0$ is not self evident. There is a proof of this in [10] that uses ring theory. □

This was also proven in a more general case in [12]. Yet another proof can be found in Appendix 2 of [13].

8.2 More recent results

Rivera-Letelier's paper [13] is a generalisation of Tan Lei's theorem for all parameters c_0 where the orbit 0 zero is *non-returning*, meaning that there is a neighbourhood U of 0 such that $f_{c_0}^n(0) \notin U$ for all $n \geq 1$. It also includes a bound on the variation of K_{c_0} around these parameters, *i.e.* that there exists a $C > 0$ such that

$$d_H(K_c, K_{c_0}) < C|c - c_0|^{1/2}$$

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