# 4 Continuity, Part I (On a Point)

## 4.1 Definition of continuity

A function $f$ is **continuous** at $a$ when

$$\lim_{x \to a} f(x) = f(a)$$

Inlining the limits definition, $f$ is continuous at $a$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$.

We can simplify this definition slightly. Observe that in continuous functions $f(a)$ exists, and at $x = a$ we get $f(x) - f(a) = 0$. Thus we can relax the constraint $0 < |x - a| < \delta$ to $|x - a| < \delta$.

A function $f$ is **continuous on an interval** $(a, b)$ if it's continuous at all $c \in (a, b)$[3].

#### Nonzero Neighborhood Lemma

Armed with these definitions we can extend the half-value neighborhood lemma (see 2.1) in a useful way. The *nonzero neighborhood lemma* will come in handy when we prove the intermediate value theorem (see 6.1), so we may as well prove the lemma now.

Suppose $f$ is continuous at $a$, and $f(a) \neq 0$. Then there exists $\delta > 0$ such that:

1. if $f(a) < 0$ then $f(x) < 0$ for all $x$ in $|x - a| < \delta$.
2. if $f(a) > 0$ then $f(x) > 0$ for all $x$ in $|x - a| < \delta$.

*Intuitively* the lemma states that there is some interval around $a$ on which $f(x) \neq 0$ and has the same sign as $f(a)$.

**Proof.** The proof follows trivially from the half-value neighborhood lemma.

## 4.2 Recognizing continuous functions

The following theorems allow us to tell at a glance that large classes of functions are continuous (e.g. polynomials, rational functions, etc.)

#### Five easy proofs

**Constants.** Let $f(x) = c$. Then $f$ is continuous at all $a$ because

$$\lim_{x \to a} f(x) = c = f(a)$$

[3] Closed intervals are a tiny bit harder, and I'm keeping them out for brevity.

**Identity.** Let $f(x) = x$. Then $f$ is continuous at all $a$ because

$$\lim_{x \to a} f(x) = a = f(a)$$

**Addition.** Let $f, g \in \mathbf{R} \to \mathbf{R}$ be continuous at $a$. Then $f + g$ is continuous at $a$ because

$$\lim_{x \to a} (f + g)(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = f(a) + g(a) = (f + g)(a)$$

**Multiplication.** Let $f, g \in \mathbf{R} \to \mathbf{R}$ be continuous at $a$. Then $f \cdot g$ is continuous at $a$ because

$$\lim_{x \to a} (fg)(x) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = f(a) \cdot g(a) = (fg)(a)$$

**Reciprocal.** Let $g$ be continuous at $a$. Then $\frac{1}{g}$ is continuous at $a$ where $g(a) \neq 0$ because

$$\lim_{x \to a} \left(\frac{1}{g}\right)(x) = \frac{1}{\lim_{x \to a} g(x)} = \frac{1}{g(a)} = \left(\frac{1}{g}\right)(a)$$

**Slightly harder proof: composition**

Let $f, g \in \mathbf{R} \to \mathbf{R}$. Let $g$ be continuous at $a$, and let $f$ be continuous at $g(a)$. Then $f \circ g$ is continuous at $a$. Put differently, we want to show

$$\lim_{x \to a} (f \circ g)(x) = (f \circ g)(a)$$

Unpacking the definitions, let $\epsilon > 0$ be given. We want to show there exists $\delta > 0$ such that $|x - a| < \delta$ implies

$$\begin{aligned} &|(f \circ g)(x) - (f \circ g)(a)| \\ &\qquad = |f(g(x)) - f(g(a))| < \epsilon \end{aligned}$$

By problem statement we have two continuities.

**First**, $f$ is continuous at $g(a)$, i.e. $\lim_{X \to g(a)} f(X) = f(g(a))$. Thus there exists $\delta' > 0$ such that $|X - g(a)| < \delta'$ implies $|f(X) - f(g(a))| < \epsilon$.

**Second**, $g$ is continuous at $a$, i.e. $\lim_{x \to a} g(x) = g(a)$. Thus there exists $\delta > 0$ such that $|x - a| < \delta$ implies $|g(x) - g(a)| < \epsilon$. Since we can make $\epsilon$ be anything, we can set it to $\delta'$.

I.e. there exists $\delta > 0$ such that $|x - a| < \delta$ implies $|g(x) - g(a)| < \delta'$. Intuitively, $g(x)$ is close to $g(a)$. But by the first continuity, any $X$ close to $g(a)$ implies

$$|f(X) - f(g(a))| < \epsilon$$

Thus $|f(g(x)) - f(g(a))| < \epsilon$, as desired.

## 4.3 Example: Stars over Babylon

Stars over Babylon is a modification of the Dirichlet function (see 3.1), defined as follows:

$$f(x) = \begin{cases} 0, & x \text{ irrational}, 0 < x < 1 \\ 1/q, & x = p/q \text{ in lowest terms}, 0 < x < 1. \end{cases}$$

**Claim:** for $0 < a < 1$, $\lim_{x \to a} f(x) = 0$.

**Proof.** Let $\epsilon > 0$ be given. We must find $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - 0| < \epsilon$. For *any* $\delta > 0$, $0 < |x - a| < \delta$ implies one of two cases for all $x$: either $x$ is irrational or it is rational.

If $x$ is irrational, $|f(x) - 0| = 0 < \epsilon$.

Otherwise, if $x = p/q$ in the lowest terms is rational, $f(x) = 1/q$. Let $n \in \mathcal{N}$ such that $1/n < \epsilon$. We will look for $\delta$ such that:

$$f\left(\frac{p}{q}\right) = \frac{1}{q} < \frac{1}{n} < \epsilon$$

Observe that when $q > n$, $f(\frac{p}{q}) = \frac{1}{q} < \frac{1}{n}$. Thus the only rationals that *could* result in $f(\frac{p}{q}) \geq 1/n$ are ones where $q \leq n$:

$$A = \{\frac{1}{2};\quad \frac{1}{3}, \frac{2}{3};\quad \frac{1}{4}, \frac{3}{4};\quad \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5},\quad \ldots,\quad \frac{1}{n}, \ldots, \frac{n-1}{n}\}$$

This set has a finite length, and thus *one* $p/q \in A$ is closest to $a$. Fix $\delta = |a - p/q|$ (i.e. anything less than this distance). This guarantees $0 < |x - a| < \delta$ implies $x \notin A$ for all $x$, and thus $f(x) < 1/n < \epsilon$ for all $x$, as desired.

**Claim:** $f(x)$ is continuous at all irrationals, discontinuous at all rationals.

**Proof:** we've just proven for $0 < a < 1$, $\lim_{x \to a} f(x) = 0$. By definition $f(x)$ is zero for all irrationals, and nonzero for all rationals. Thus $\lim_{x \to a} f(x) = f(x)$ for all irrationals, and $\lim_{x \to a} f(x) \neq f(x)$ for all rationals.