2 Limits, Part I (Blessed Path)

2.1 Formal limits definition

Definition: $\lim_{x \to a} f(x) = L$ when for any $\epsilon \in \mathbf{R}$ there exists $\delta \in \mathbf{R}$ such that for all x, $0 < |x - a| < \delta$ implies $|f(x) - L| < \epsilon$. (Also $\epsilon > 0, \delta > 0$.)

Here is what this says. Suppose $\lim_{x \to a} f(x) = L$. You pick any interval on the y-axis around L. Make it as small (or as large) as you want. I'll produce an interval on the x-axis around a. You can take any number from my interval, plug it into f, and the output will stay within the bounds you specified.

So ϵ specifies the distance away from L along the y-axis, and δ specifies the distance away from a along the x-axis. Take any x within δ of a, plug it into f, and the result is guaranteed to be within ϵ of L. $\lim_{x \to a} f(x) = L$ just means there exists such δ for any ϵ.

Limit uniqueness

Suppose $\lim_{x \to a} f(x) = L$. It's easy to assume L is the only limit around a, but such a thing needs to be proved. We prove this here. More formally, suppose $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} f(x) = M$. We prove that $L = M$.

Suppose for contradiction $L \neq M$. Assume without loss of generality $L > M$. By limit definition, for all $\epsilon > 0$ there exists a positive $\delta \in \mathbf{R}$ such that $0 < |x - a| < \delta$ implies

- $|f(x) - L| < \epsilon \implies L - \epsilon < f(x)$
- $|f(x) - M| < \epsilon \implies f(x) < M + \epsilon$

for all x. Thus

$$\begin{aligned} &L - \epsilon < f(x) < M + \epsilon \\ &\implies L - \epsilon < M + \epsilon \\ &\implies L - M < 2\epsilon \end{aligned}$$

The above is true for all ϵ. Now let's narrow our attention and consider a concrete $\epsilon = (L - M)/4$, which we easily find leads to a contradiction[2]:

$$\begin{aligned} &L - M < 2\epsilon && \\ &\implies (L - M)/4 < \epsilon/2 && \text{dividing both sides by 4} \\ &\implies \epsilon < \epsilon/2 && \text{recall we set } \epsilon = (L - M)/4 \end{aligned}$$

We have a contradiction, and so $L = M$ as desired.

[2] note we assumed $L > M$, thus $\epsilon = (L - M)/4 > 0$

Half-Value Neighborhood Lemma

This lemma will come in handy later, so we may as well prove it now. Suppose $M \neq 0$ and $\lim_{x \to a} g(x) = M$. We show that there exists some δ such that $0 < |x - a| < \delta$ implies $|g(x)| \geq |M|/2$ for all x.

Intuitively, the lemma states the following: when a function g approaches a nonzero limit M near a point, there exists an interval in which the values of g are closer to M than to zero.

Proof. The claim that $|g(x)| \geq |M|/2$ is equivalent to

$$g(x) \leq -|M|/2 \quad \text{or} \quad g(x) \geq |M|/2$$

There are two possibilities: either $M > 0$ or $M < 0$. Let's consider each possibility separately.

Case 1. Suppose $M > 0$. Then to show $|g(x)| \geq |M|/2$ it is sufficient to show *either* $g(x) \leq -M/2$ or $g(x) \geq M/2$. We will show $g(x) \geq M/2$. Fix $\epsilon = M/2$. By limit definition there is some δ such that $0 < |x - a| < \delta$ implies for all x

$$\begin{aligned}
&|g(x) - M| < M/2 && \\
&\implies -M/2 < g(x) - M && \\
&\implies M/2 < g(x) && \text{add } M \text{ to both sides} \\
&\implies g(x) > M/2 && \text{note } \geq \text{ is correct but not tight}
\end{aligned}$$

Case 2. Suppose $M < 0$. We must show either $g(x) \leq M/2$ or $g(x) \geq -M/2$. We will show $g(x) \leq M/2$. Fix $\epsilon = -M/2$. Then

$$\begin{aligned}
&|g(x) - M| < -M/2 && \\
&\implies g(x) - M < -M/2 && \\
&\implies g(x) < M/2 && \text{add } M \text{ to both sides;} \\
& && \text{note } \leq \text{ is correct but not tight}
\end{aligned}$$

QED.

2.2 Evaluation mechanics proofs

Armed with the formal definition, we can use it to rigorously prove the five theorems useful for evaluating limits (constants, identity, addition, multiplication, reciprocal). Let's do that now.

Constants

Let $f(x) = c$. We prove that $\lim_{x \to a} f(x) = c$ for all a.

Let $\epsilon > 0$ be given. Pick any positive δ. Then for all x such that $0 < |x - a| < \delta$, $|f(x) - c| = |c - c| = 0 < \epsilon$. QED.

(Note that we can pick any positive $\delta > 0$, e.g. $1, 10, \frac{1}{10}$.)

Identity

Let $f(x) = x$. We prove that $\lim_{x \to a} f(x) = a$ for all a.

Let $\epsilon > 0$ be given. We need to find $\delta > 0$ such that for all x in $0 < |x - a| < \delta$, $|f(x) - a| = |x - a| < \epsilon$. I.e. we need to find a δ such that $|x - a| < \delta$ implies $|x - a| < \epsilon$. This obviously works for any $\delta \leq \epsilon$. QED.

(Note the many options for δ, e.g. $\delta = \epsilon$, $\delta = \frac{\epsilon}{2}$, etc.)

Addition

Let $f, g \in \mathbf{R} \to \mathbf{R}$. We prove that

$$\lim_{x \to a} (f + g)(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

Let $L_f = \lim_{x \to a} f(x)$ and let $L_g = \lim_{x \to a} g(x)$. Let $\epsilon > 0$ be given. We must show there exists $\delta > 0$ such that for all x bounded by $0 < |x - a| < \delta$ the following inequality holds:

$$|(f + g)(x) - (L_f + L_g)| < \epsilon$$

I.e. we're trying to show $\lim_{x \to a}(f+g)(x)$ equals to $L_f + L_g$, the sum of the other two limits. Let's convert the left side of this inequality into a more convenient form:

$$\begin{aligned}
|(f + g)(x) - (L_f + L_g)| &= |f(x) + g(x) - (L_f + L_g)| \\
&= |(f(x) - L_f) + (g(x) - L_g)| \\
&\leq |(f(x) - L_f)| + |(g(x) - L_g)| \quad \text{by triangle inequality}
\end{aligned}$$

By limit definition there exist positive δ_f, δ_g such that for all x

- $0 < |x - a| < \delta_f$ implies $|f(x) - L_f| < \epsilon/2$
- $0 < |x - a| < \delta_g$ implies $|g(x) - L_g| < \epsilon/2$

Recall that we can make ϵ as small as we like. Here we pick deltas for $\epsilon/2$ because it's convenient to make the equations work, as you will see in a second. For all x bounded by $0 < |x - a| < \min(\delta_f, \delta_g)$ we have

$$|(f(x) - L_f)| < \epsilon/2 \quad \text{and} \quad |(g(x) - L_g)| < \epsilon/2$$

Fix $\delta = \min(\delta_f, \delta_g)$. Then for all x bounded by $0 < |x - a| < \delta$ we have

$$\begin{aligned}|(f+g)(x)-(L_f+L_g)| &\le |(f(x)-L_f)|+|(g(x)-L_g)| \\ &< \epsilon/2+\epsilon/2=\epsilon\end{aligned}$$

as desired.

Multiplication

Let $f, g \in \mathbf{R} \to \mathbf{R}$. We prove that

$$\lim_{x\to a}(fg)(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$$

Let $L_f = \lim_{x\to a} f(x)$ and let $L_g = \lim_{x\to a} g(x)$. Let $\epsilon > 0$ be given. We must show there exists $\delta > 0$ such that for all x bounded by $0 < |x-a| < \delta$ the following inequality holds:

$$|(fg)(x) - (L_f L_g)| < \epsilon$$

(i.e. we're trying to show $\lim_{x\to a}(fg)(x)$ equals to $L_f L_g$, the product of the other two limits.) Let's convert the left side of this inequality into a more convenient form:

$$\begin{aligned}|(fg)(x)-(L_fL_g)| &= |f(x)g(x)-L_fL_g| \\ &= |f(x)g(x)-L_fg(x)+L_fg(x)-L_fL_g| \\ &= |g(x)(f(x)-L_f)+L_f(g(x)-L_g)| \\ &\le |g(x)(f(x)-L_f)|+|L_f(g(x)-L_g)| && \text{by triangle inequality} \\ &= |g(x)||f(x)-L_f|+|L_f||g(x)-L_g| && \text{in general } |ab|=|a||b|\end{aligned}$$

We now need to show there exists δ such that $0 < |x-a| < \delta$ implies

$$|g(x)||f(x)-L_f| + |L_f||g(x)-L_g| < \epsilon$$

We will do that by finding δ such that

1. $|g(x)||f(x)-L_f| < \epsilon/2$
2. $|L_f||g(x)-L_g| < \epsilon/2$

First, we show $|g(x)||f(x)-L_f| < \epsilon/2$.

By limit definition we can find δ_1 to make $|f(x)-L_f|$ as small as we like. But how small? To make $|g(x)||f(x)-L_f| < \epsilon/2$ we must find a delta such that $|f(x)-L_f| < \epsilon/2g(x)$. But to do that we need to get a bound on $g(x)$. Fortunately we know there exists δ_2 such that $|g(x)-L_g| < 1$ (we pick 1 because we must pick some bound, and 1 is as good as any). Thus $|g(x)| < |L_g|+1$. And so, we can pick δ_1 such that $|f(x)-L_f| < \epsilon/2(|L_g|+1)$.

Second, we show $|L_f||g(x) - L_g| < \epsilon/2$.

That is easy. By limit definition there exists a δ_3 such that $0 < |x - a| < \delta_3$ implies $|g(x) - L_g| < \epsilon/2|L_f|$ for all x. Actually, we need a δ_3 such that $0 < |x - a| < \delta_3$ implies $|g(x) - L_g| < \frac{\epsilon}{2(|L_f|+1)}$ for all x to avoid divide by zero, and of course that exists too.

Fix $\delta = \min(\delta_1, \delta_2, \delta_3)$. Now

$$\begin{aligned}|(fg)(x) - (L_f L_g)| &\leq |g(x)||f(x) - L_f| + |L_f||g(x) - L_g| \\ &< e/2 + e/2 = e\end{aligned}$$

as desired.

Reciprocal

Let $\lim_{x \to a} f(x) = L$. We prove $\lim_{x \to a} \left(\frac{1}{f}\right)(x) = 1/L$ when $L \neq 0$.

First we show $\frac{1}{f}$ is defined near a. By half-value neighborhood lemma (see 2.1) there exists δ_1 such that $0 < |x - a| < \delta_1$ implies $|f(x)| \geq |L|/2$ where $L \neq 0$. Therefore $f(x) \neq 0$ near a, and thus $\frac{1}{f}$ near a is defined.

Now all we must do is find a delta such that $\left|\frac{1}{f}(x) - \frac{1}{L}\right| < \epsilon$. Let's make the equation more convenient:

$$\begin{aligned}\left|\frac{1}{f}(x) - \frac{1}{L}\right| &= \left|\frac{1}{f(x)} - \frac{1}{L}\right| \\ &= \left|\frac{L - f(x)}{Lf(x)}\right| \\ &= \frac{|f(x) - L|}{|L||f(x)|} \\ &= \frac{|f(x) - L|}{|L|} \cdot \frac{1}{|f(x)|}\end{aligned}$$

Above we showed there exists δ_1 such that $0 < |x - a| < \delta_1$ implies $|f(x)| \geq |L|/2$. Raising both sides to -1 we get $|\frac{1}{f(x)}| \leq \frac{2}{|L|}$. Continuing the chain of reasoning above we get

$$\begin{aligned}\frac{|f(x) - L|}{|L|} \cdot \frac{1}{|f(x)|} &\leq \frac{|f(x) - L|}{|L|} \cdot \frac{2}{|L|} \\ &= \frac{2}{|L|^2}|f(x) - L|\end{aligned}$$

(if you're confused about why this inequality works, left-multiply both sides of $|\frac{1}{f(x)}| \leq \frac{2}{|L|}$ by $\frac{|f(x)-L|}{|L|}$.) Thus we must find δ_2 such that

$$\frac{2}{|L|^2}|f(x) - L| < \epsilon$$

That is easy. Since $\lim_{x \to a} f(x) = L$ we can make $|f(x) - L|$ as small as we like. Dividing both sides by $\frac{2}{|L|^2}$, we must make $|f(x) - L| < \frac{|L|^2\epsilon}{2}$. Thus we must fix $\delta = \min(\delta_1, \delta_2)$. QED.

2.3 Low-level proofs

While high level theorems allow us to easily compute complicated limits, it's instructive to compute a few limits for complicated functions straight from the definition. We do that here.

Limits of quadratic functions

We will prove directly from the limits definition that $\lim_{x \to a} x^2 = a^2$. Let $\epsilon > 0$ be given. We must show there exists δ such that $|x^2 - a^2| < \epsilon$ for all x in $0 < |x - a| < \delta$.

Observe that

$$|x^2 - a^2| = |(x - a)(x + a)| = |x - a||x + a|$$

Thus we must pick δ such that $|x - a||x + a| < \epsilon$. Since $0 < |x - a| < \delta$, picking δ conveniently happens to bound $|x - a|$, letting us make it as small as we want. But to know how small, we need to find an upper bound on $|x + a|$. We can do it as follows.

Pick an arbitrary $\delta = 1$ (we may pick any arbitrary delta, e.g. 1/10, 10, etc.) Then since $|x - a| < \delta$:

$$\begin{aligned} &|x - a| < 1 \\ &\implies -1 < x - a < 1 \\ &\implies 2a - 1 < x + a < 2a + 1 \qquad \text{add } 2a \text{ to both sides} \end{aligned}$$

We now have a bound on $x + a$, but we need one on $|x + a|$. It's easy to see $|x + a| < \max(|2a - 1|, |2a + 1|)$. By triangle inequality ($|a + b| \leq |a| + |b|$):

$$\begin{aligned} |2a - 1| &\leq |2a| + |-1| = |2a| + 1 \\ |2a + 1| &\leq |2a| + |1| = |2a| + 1 \end{aligned}$$

Thus $|x + a| < |2a| + 1$, provided $|x - a| < 1$. Coming back to our original goal, $|x - a||x + a| < \epsilon$ when

- $|x - a| < 1$ and
- $|x - a| < \frac{\epsilon}{|2a|+1}$

Putting these together, $\delta = \min(1, \frac{\epsilon}{|2a|+1})$.

Limits of fractions

We will prove directly from the limits definition that $\lim_{x\to 2} \frac{3}{x} = \frac{3}{2}$. Let $\epsilon > 0$ be given. We must show there exists $\delta > 0$ such that $|\frac{3}{x} - \frac{3}{2}| < \epsilon$ for all x in $0 < |x-2| < \delta$.

Let's manipulate $|\frac{3}{x} - \frac{3}{2}|$ to make it more convenient:

$$\left|\frac{3}{x} - \frac{3}{2}\right| = \left|\frac{6-3x}{2x}\right| = \frac{3}{2}\frac{|x-2|}{|x|}$$

Thus we need to find δ such that

$$\begin{aligned} &\frac{3}{2}\frac{|x-2|}{|x|} < \epsilon \\ &\implies \frac{|x-2|}{|x|} < \frac{2\epsilon}{3} \end{aligned}$$

Conveniently $0 < |x-2| < \delta$ bounds $|x-2|$. But now we need to find a bound for $|x|$. It would be extra convenient if we could show $|x| > 1$. Then we could set $\delta = \frac{2\epsilon}{3}$ (and thus bound $|x-2| < \frac{2\epsilon}{3}$). A denominator greater than 1 would only make the fraction smaller than $\frac{2\epsilon}{3}$, ensuring $\frac{|x-2|}{|x|} < \frac{2\epsilon}{3}$ holds.

We will do exactly that. Pick an arbitrary $\delta = 1$ (we may pick any arbitrary delta, e.g. 1/10, 10, etc.) Then since $|x-2| < \delta$

$$\begin{aligned} &|x-2| < 1 \\ &\implies -1 < x-2 < 1 \\ &\implies 1 < x < 3 \\ &\implies 1 < |x| < 3 \end{aligned}$$

Yes!! Luckily $\delta = 1$ implies $|x| > 1$! Thus, provided that $|x-2| < 1$ and $|x-2| < \frac{2\epsilon}{3}$, the inequality $|\frac{3}{x} - \frac{3}{2}| < \epsilon$ holds. Putting the two constraints together, we get $\delta = \min(1, \frac{2\epsilon}{3})$.